Proceedings of Conference on Function Algebras 2017

March 2018

Conference on Function Algebras 2017 was held at School of Pharmacy, Nihon University during the period from October 20 to 22, 2017. Last year, four mathematicians from Korean universities participated. This year, in addition to Professor Boo Rim Choe, Professor Young Joo Lee, and Jongho Yang from the Republic of Korea, Professor Lajos Molnár from Hungary participated. In addition, there were one speaker and several participants from fields other than Function Algebras. I hope participants will monotone increase in the future.

I apologize that I was late for completing the proceedings. I hope that this proceedings will be useful for your research.

Organizer : Norio Niwa (Nihon University)

Conference on Function Algebras 2017

October 22 (Sun)

On a counterpart to the Riesz representation theorem in the Möbius gyrovector space

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1　導入

Ungar の (real inner product) gyrovector space では交換法則, 結合法則, 分配法則がそのままでは成り立たないにもかかわらず, 最近の研究によって, 或いは定義から当たり前にというべきか, Möbius gyrovector space については Hilbert 空間との間に強いアナロジーがはたらく事が知られてきた. 閉部分空間による直交分解, 閉部分空間さらには閉凸集合の最近点, 正規直交基底による直交展開などである. 線形作用素の対応物はどうなっているだろうかという自然な問題について, 分かっていることを報告する.

この節では Möbius Gyrovector Space の定義および若干の注意事項と, 直交基底に関する任意元の直交ジャイロ展開までの結果のうち, いくつかの主要なものを述べる. 抽象的な (gyrocommutative) gyrogroup, gyrovector space の定義や基本的事項は [U] を参照していただきたい. なお, 結果の一部は阿部敏一氏（茨城大学）との共同研究によるものです.

Möbius Gyrovector Spaces.[U] $\mathbb{V}$ を任意の実内積空間, 固定された正の数 s に対して

$$\mathbb{V}_s = \{\boldsymbol{a} \in \mathbb{V};\ ||\boldsymbol{a}|| < s\}$$

とする. Möbius の和および Möbius のスカラー倍は

$$\boldsymbol{a} \oplus_{\mathrm{M}} \boldsymbol{b} = \frac{\left(1 + \frac{2}{s^2}\boldsymbol{a}\cdot\boldsymbol{b} + \frac{1}{s^2}||\boldsymbol{b}||^2\right)\boldsymbol{a} + \left(1 - \frac{1}{s^2}||\boldsymbol{a}||^2\right)\boldsymbol{b}}{1 + \frac{2}{s^2}\boldsymbol{a}\cdot\boldsymbol{b} + \frac{1}{s^4}||\boldsymbol{a}||^2||\boldsymbol{b}||^2}$$

$$r \otimes_{\mathrm{M}} \boldsymbol{a} = s\ \tanh\left(r \tanh^{-1}\frac{||\boldsymbol{a}||}{s}\right)\frac{\boldsymbol{a}}{||\boldsymbol{a}||} \quad (\text{if } \boldsymbol{a} \neq \boldsymbol{0}), \qquad r \otimes_{\mathrm{M}} \boldsymbol{0} = \boldsymbol{0}$$

for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}_s$, $r \in \mathbb{R}$ によって定義される.
公理 (VV) の, $||\mathbb{V}_s|| = (-s, s)$ における演算 $\oplus_{\mathrm{M}}$, $\otimes_{\mathrm{M}}$（同一の記号が使われる）は

$$a \oplus_{\mathrm{M}} b = \frac{a + b}{1 + \frac{1}{s^2}ab}$$

$$r \otimes_{\mathrm{M}} a = s \tanh\left(r \tanh^{-1}\frac{a}{s}\right)$$

for all $a, b \in (-s, s)$, $r \in \mathbb{R}$ によって定義される.

このとき, $(\mathbb{V}_s, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$ は gyrovector space となる. $\oplus_{\mathrm{M}}, \otimes_{\mathrm{M}}$ をそれぞれ単に $\oplus, \otimes$ と書く.

異なる種類の演算が同一の数式に現れたならば, (1) 通常のスカラー倍 (2) 演算 $\otimes$ (3) 演算 $\oplus$ で優先順を与える, すなわち,

$$r_1 \otimes w_1 \boldsymbol{a}_1 \oplus r_2 \otimes w_2 \boldsymbol{a}_2 = \{r_1 \otimes (w_1 \boldsymbol{a}_1)\} \oplus \{r_2 \otimes (w_2 \boldsymbol{a}_2)\}.$$

そしてこのような場合の括弧は省略する.

一般には, 演算は可換でも, 結合的でも, 分配的でもないことに注意する:

$$\boldsymbol{a} \oplus \boldsymbol{b} \neq \boldsymbol{b} \oplus \boldsymbol{a}$$

$$\boldsymbol{a} \oplus (\boldsymbol{b} \oplus \boldsymbol{c}) \neq (\boldsymbol{a} \oplus \boldsymbol{b}) \oplus \boldsymbol{c}$$

$$r \otimes (\boldsymbol{a} \oplus \boldsymbol{b}) \neq r \otimes \boldsymbol{a} \oplus r \otimes \boldsymbol{b}$$

$$t(\boldsymbol{a} \oplus \boldsymbol{b}) \neq t\boldsymbol{a} \oplus t\boldsymbol{b}.$$

しかし, 左（および右）ジャイロ結合法則, ジャイロ交換法則, スカラー分配法則, スカラー結合法則などがあるように, gyrovector space は解明すべき豊かな対称性を有している.

$s \to \infty$ とすると $\mathbb{V}_s$ は全空間 $\mathbb{V}$ に拡大し, 演算 $\oplus, \otimes$ は通常のベクトル和, スカラー倍に近づく.

命題. [U]

$$\boldsymbol{a} \oplus \boldsymbol{b} \to \boldsymbol{a} + \boldsymbol{b} \quad (s \to \infty)$$

$$r \otimes \boldsymbol{a} \to r\boldsymbol{a} \quad (s \to \infty).$$

Definition. A subset M of $\mathbb{V}_s$ is a gyrovector subspace if

$$\boldsymbol{a}, \boldsymbol{b} \in M,\ r \in \mathbb{R} \quad \Rightarrow \quad \boldsymbol{a} \oplus \boldsymbol{b} \in M,\ r \otimes \boldsymbol{a} \in M.$$

For any subset A of $\mathbb{V}_s$, we use the notation

$$\bigvee^g A = \bigcap \{M;\ A \subset M,\ M \text{ is a gyrovector subspace of } \mathbb{V}_s\}.$$

Theorem.(Abe and W) Let $(\mathbb{V}_1, \oplus, \otimes)$ be the Möbius gyrovector space, $\mathbf{0} \neq \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n \in \mathbb{V}_1$ and let $(i_1, \cdots, i_n)$ be a permutation of $(1, \cdots, n)$. For an arbitrary given order of gyroaddition of $r_{i_1} \otimes \boldsymbol{a}_{i_1} \oplus \cdots \oplus r_{i_n} \otimes \boldsymbol{a}_{i_n}$,

$$\bigvee^g \{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\} = \{r_{i_1} \otimes \boldsymbol{a}_{i_1} \oplus \cdots \oplus r_{i_n} \otimes \boldsymbol{a}_{i_n};\ r_{i_1}, \cdots, r_{i_n} \in \mathbb{R}\}$$

$$= \left\{ t_1 \frac{\boldsymbol{a}_1}{||\boldsymbol{a}_1||} + \cdots + t_n \frac{\boldsymbol{a}_n}{||\boldsymbol{a}_n||};\ t_1, \cdots, t_n \in \mathbb{R} \right\} \cap \mathbb{V}_1.$$

Remark. We have the same result for finitely generated gyrovector subspaces in the Einstein gyrovector space.

Theorem.(Abe and W) Let $\mathbb{V}$ be a real Hilbert space and let $(\mathbb{V}_1, \oplus, \otimes)$ be the Möbius gyrovector space, and let M be a gyrovector subspace of $\mathbb{V}_1$ that is topologically relatively closed. Suppose that

$$\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_2, \quad \boldsymbol{x}_1 \in \mathrm{clin} M, \ \ \boldsymbol{x}_2 \in M^{\perp}$$

is the (ordinary) orthogonal decomposition of an arbitrary element $\boldsymbol{x} \in \mathbb{V}_1$ with respect to $\mathrm{clin} M$, which is the closed linear subspace generated by M. Then, a unique pair $(\boldsymbol{y}, \boldsymbol{z})$ exists that satisfies

$$\boldsymbol{x} = \boldsymbol{y} \oplus \boldsymbol{z}, \quad \boldsymbol{y} \in M, \ \ \boldsymbol{z} \in M^{\perp} \cap \mathbb{V}_1.$$

Moreover, if $\boldsymbol{x}_1, \boldsymbol{x}_2 \neq \boldsymbol{0}$, then these elements $\boldsymbol{y}, \boldsymbol{z}$ are determined by

$$\boldsymbol{y} = \lambda_1 \boldsymbol{x}_1, \ \ \boldsymbol{z} = \lambda_2 \boldsymbol{x}_2,$$

where

$$\lambda_1 = \frac{||\boldsymbol{x}_1||^2 + ||\boldsymbol{x}_2||^2 + 1 - \sqrt{(||\boldsymbol{x}_1||^2 + ||\boldsymbol{x}_2||^2 + 1)^2 - 4||\boldsymbol{x}_1||^2}}{2||\boldsymbol{x}_1||^2}$$

$$\lambda_2 = \frac{||\boldsymbol{x}_1||^2 + ||\boldsymbol{x}_2||^2 - 1 + \sqrt{(||\boldsymbol{x}_1||^2 + ||\boldsymbol{x}_2||^2 + 1)^2 - 4||\boldsymbol{x}_1||^2}}{2||\boldsymbol{x}_2||^2}.$$

In addition, the inequalities $0 < \lambda_1 < 1$ and $\lambda_2 > 1$ hold.

Definition.[U] The Möbius gyrodistance function d on a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ is defined by the equation

$$d(\boldsymbol{a}, \boldsymbol{b}) = ||\boldsymbol{b} \ominus \boldsymbol{a}||.$$

Moreover, the Poincaré distance function h on the ball $\mathbb{V}_s$ is introduced by the equation

$$h(\boldsymbol{a}, \boldsymbol{b}) = \tanh^{-1} \frac{d(\boldsymbol{a}, \boldsymbol{b})}{s}$$

for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}_s$. Then h satisfies the triangle inequality, so that $(\mathbb{V}_s, h)$ is a metric space. It is also complete as a metric space provided $\mathbb{V}$ is complete.

Definition. Let $\{\boldsymbol{a}_n\}_n$ be a sequence in $\mathbb{V}_s$. We say that a series

$$\Big(((\boldsymbol{a}_1 \oplus \boldsymbol{a}_2) \oplus \boldsymbol{a}_3) \oplus \cdots \oplus \boldsymbol{a}_n\Big) \oplus \cdots$$

converges if there exists an element $\boldsymbol{x} \in \mathbb{V}_s$ such that $h(\boldsymbol{x}, \boldsymbol{x}_n) \to 0 \ \ (n \to \infty)$, where the sequence $\{\boldsymbol{x}_n\}_n$ is defined recursively by $\boldsymbol{x}_1 = \boldsymbol{a}_1$ and $\boldsymbol{x}_n = \boldsymbol{x}_{n-1} \oplus \boldsymbol{a}_n$. In this case, we say the series converges to $\boldsymbol{x}$ and denote

$$\boldsymbol{x} = \Big(((\boldsymbol{a}_1 \oplus \boldsymbol{a}_2) \oplus \boldsymbol{a}_3) \oplus \cdots \oplus \boldsymbol{a}_n\Big) \oplus \cdots .$$

Theorem. Let $\{\boldsymbol{e}_n\}_{n=1}^{\infty}$ be a complete orthonormal sequence in a real Hilbert space $\mathbb{V}$. Let $\{w_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ such that $0 < w_n < s$ for all n. Then, any $\boldsymbol{x} \in \mathbb{V}_s$ can be expressed as a form of orthogonal gyroexpansion

$$\boldsymbol{x} = r_1 \otimes w_1 \boldsymbol{e}_1 \oplus r_2 \otimes w_2 \boldsymbol{e}_2 \oplus \cdots \oplus r_n \otimes w_n \boldsymbol{e}_n \oplus \cdots ,$$

where the sequence of gyrocoefficients $\{r_n\}_{n=1}^{\infty}$ is determined by the following equations:

$$x_n = \boldsymbol{x} \cdot \boldsymbol{e}_n, \qquad \boldsymbol{x}_n^{(1)} = \sum_{j=1}^{n} x_j \boldsymbol{e}_j, \qquad \boldsymbol{x}_n^{(2)} = \sum_{j=n+1}^{\infty} x_j \boldsymbol{e}_j$$

$$\boldsymbol{u}_j = \mu_{j-1}^{(2)} \cdots \mu_1^{(2)} x_j \boldsymbol{e}_j \ \ (j = 2, 3, \cdots) \qquad \boldsymbol{u}_1 = x_1 \boldsymbol{e}_1 = \boldsymbol{x}_1^{(1)}$$

$$\boldsymbol{v}_j = \mu_{j-1}^{(2)} \cdots \mu_1^{(2)} \boldsymbol{x}_j^{(2)} \ \ (j = 2, 3, \cdots) \qquad \boldsymbol{v}_1 = \boldsymbol{x}_1^{(2)}$$

$$\mu_j^{(1)} = \frac{||\boldsymbol{u}_j||^2 + ||\boldsymbol{v}_j||^2 + s^2 - \sqrt{(||\boldsymbol{u}_j||^2 + ||\boldsymbol{v}_j||^2 + s^2)^2 - 4s^2||\boldsymbol{u}_j||^2}}{2||\boldsymbol{u}_j||^2}$$

$$\mu_j^{(2)} = \frac{||\boldsymbol{u}_j||^2 + ||\boldsymbol{v}_j||^2 - s^2 + \sqrt{(||\boldsymbol{u}_j||^2 + ||\boldsymbol{v}_j||^2 + s^2)^2 - 4s^2||\boldsymbol{u}_j||^2}}{2||\boldsymbol{v}_j||^2}$$

$$r_j = \frac{\tanh^{-1} \frac{\mu_j^{(1)} \mu_{j-1}^{(2)} \cdots \mu_1^{(2)} x_j}{s}}{\tanh^{-1} \frac{w_j}{s}}.$$

2　最近の結果の概要

次の 2 つの結果は, Möbius gyrovector space における Riesz の表現定理の最適な対応物にはなっていないと思う.

Theorem. Let $\mathbb{V}$ be a real Hilbert space with $\dim \mathbb{V} \geq 2$. If $f : \mathbb{V}_1 \to (-1, 1)$ satisfies that

$$f(\boldsymbol{x} \oplus \boldsymbol{y}) = f(\boldsymbol{x}) \oplus f(\boldsymbol{y})$$

$$f(r \otimes \boldsymbol{x}) = r \otimes f(\boldsymbol{x})$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}_1$, $r \in \mathbb{R}$, then we have $f \equiv 0$.

Proposition. Let $\mathbb{V}$ be a real Hilbert space. If $f : \mathbb{V} \to \mathbb{R}$ is a continuous map and satisfies that

$$f(\boldsymbol{x} \oplus_s \boldsymbol{y}) - \{f(\boldsymbol{x}) \oplus_s f(\boldsymbol{y})\} \to 0 \ \ (s \to \infty)$$

$$f(r \otimes_s \boldsymbol{x}) - r \otimes_s f(\boldsymbol{x}) \to 0 \ \ (s \to \infty)$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}$, $r \in \mathbb{R}$, then there exists a unique $\boldsymbol{c} \in \mathbb{V}$ such that $f(\boldsymbol{x}) = \boldsymbol{x}\cdot\boldsymbol{c}$ $(\boldsymbol{x} \in \mathbb{V})$. The converse is also true.

T. Abe once raised the following problem in an oral presentation[A].

Problem. What are mappings between gyrolinear spaces corresponding to linear mappings between linear spaces ?

For the definition of $\oplus_{\mathrm{E}}$, please refer to any bibliography such as [U].

Theorem.(Molnár and Virosztek) Let $\beta : \mathbb{R}^3_1 \to \mathbb{R}^3_1$ be a continuous map. We have β is an algebraic endomorphism with respect to the operation $\oplus_{\mathrm{E}}$, i.e., β satisfies

$$\beta(\boldsymbol{u} \oplus_{\mathrm{E}} \boldsymbol{v}) = \beta(\boldsymbol{u}) \oplus_{\mathrm{E}} \beta(\boldsymbol{v}) \quad (\boldsymbol{u},\, \boldsymbol{v} \in \mathbb{R}^3_1)$$

if and only if either (i) or (ii) of the following holds:
(i) there is an orthogonal matrix $O \in \mathrm{M}_3(\mathbb{R})$ such that $\beta(\boldsymbol{v}) = O\boldsymbol{v}$, $\boldsymbol{v} \in \mathbb{R}^3_1$
(ii) $\beta(\boldsymbol{v}) = \boldsymbol{0}$, $\boldsymbol{v} \in \mathbb{R}^3_1$.

Theorem.(Frenkel) For $n \geq 2$, continuous endomorphisms of the Einstein gyrogroup $(\mathbb{R}^n_1, \oplus_{\mathrm{E}})$ are precisely the restrictions to $\mathbb{R}^n_1$ of the orthogonal transformations of $\mathbb{R}^n$ and 0-map.

このように, 線形作用素がベクトルの加法（およびスカラー倍）を保存することからの直接の類推による, 演算 $\oplus$（および $\otimes$）をすべての元の対にわたってピッタリと保存する性質は, ある意味で強すぎる.

Definition. We denote by $\mathrm{M}_{n,m}(\mathbb{R})$ the set of all $n \times m$ matrices whose entries are real numbers.

The ordinary operation of matrices on vectors:

$$A\boldsymbol{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}$$

$$A : x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 \mapsto (a_{11}x_1 + a_{12}x_2)\boldsymbol{f}_1 + (a_{21}x_1 + a_{22}x_2)\boldsymbol{f}_2$$

Suppose that $\{\boldsymbol{e}_j\}_{j=1}^m$ (resp. $\{\boldsymbol{f}_i\}_{i=1}^n$) be an orthonormal basis in $\mathbb{U}$ (resp. $\mathbb{V}$). For an arbitrary element $\boldsymbol{x} \in \mathbb{U}_1$, we can apply the orthogonal gyroexpansion to get a unique m-tuple $(r_1, \cdots, r_m)$ of real numbers such that

$$\boldsymbol{x} = r_1 \otimes_1 \frac{\boldsymbol{e}_1}{2} \oplus_1 \cdots \oplus_1 r_m \otimes_1 \frac{\boldsymbol{e}_m}{2}.$$

Then we can define a map $f : \mathbb{U}_1 \to \mathbb{V}_1$ by the equation

$$f(\boldsymbol{x}) = (a_{11}r_1 + \cdots + a_{1m}r_m)\otimes_1 \frac{\boldsymbol{f}_1}{2} \oplus_1$$

$$\cdots \oplus_1 (a_{n1}r_1 + \cdots + a_{nm}r_m)\otimes_1 \frac{\boldsymbol{f}_n}{2}.$$

We say that f is the induced map from the matrix A.

For any map $f : \mathbb{U}_1 \to \mathbb{V}_1$ and any $s > 0$, we can define a map $f_s : \mathbb{U}_s \to \mathbb{V}_s$ by the equation

$$f_s(\boldsymbol{x}) = sf\left(\frac{\boldsymbol{x}}{s}\right) \quad (\boldsymbol{x} \in \mathbb{U}_s). \tag{1}$$

Theorem. Let $\mathbb{U}$ and $\mathbb{V}$ be two real Hilbert spaces and $A \in \mathrm{M}_{n,m}(\mathbb{R})$. If $f : (\mathbb{U}_1, \oplus_1, \otimes_1) \to (\mathbb{V}_1, \oplus_1, \otimes_1)$ is the induced map from the matrix A and f_s is defined by (1), then, for arbitrary $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{U}$ and $r \in \mathbb{R}$, we have

$$f_s(\boldsymbol{x} \oplus_s \boldsymbol{y}) \to A(\boldsymbol{x} + \boldsymbol{y})$$

$$f_s(\boldsymbol{x}) \oplus_s f_s(\boldsymbol{y}) \to A\boldsymbol{x} + A\boldsymbol{y}$$

$$f_s(r\otimes_s\boldsymbol{x}) \to Ar\boldsymbol{x}$$

$$r\otimes_s f_s(\boldsymbol{x}) \to rA\boldsymbol{x}$$

as $s \to \infty$.

Theorem. Let $\mathbb{U}$, $\mathbb{V}$, $\mathbb{W}$ are real Hilbert spaces. Let $A = (a_{ij}) \in \mathrm{M}_{n,m}(\mathbb{R})$, $B = (b_{ij}) \in \mathrm{M}_{p,n}(\mathbb{R})$. Suppose that $\{\boldsymbol{e}_j\}_{j=1}^m$, $\{\boldsymbol{f}_i\}_{i=1}^n$, $\{\boldsymbol{g}_k\}_{k=1}^p$ be an orthonormal basis in $\mathbb{U}$, $\mathbb{V}$, $\mathbb{W}$, respectively. Let f (resp. g) be the induced map from matrix A (resp. B). Then the composed map $g \circ f$ is also an induced map from the matrix BA.

Theorem. Let f be the induced map from a matrix A and f^* be the induced map from the adjoint matrix A^*. Then

$$-f_s^*(\boldsymbol{x}){\cdot}\boldsymbol{y} \oplus_s \boldsymbol{x}{\cdot}f_s(\boldsymbol{y}) \to 0 \quad (s \to \infty).$$

このように, Hilbert 空間の間の線形作用素に対応する Möbius gyrovector space 間の写像のあるクラスとして, 次が考えられる. $\mathbb{U}_1, \mathbb{V}_1$ を 2 つの Möbius gyrovector spaces, $f : \mathbb{U}_1 \to \mathbb{V}_1$ を写像とする. ある線形作用素 $T : \mathbb{U} \to \mathbb{V}$ が存在して任意の $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{U}$ と $r \in \mathbb{R}$ に対して

$$f_s(\boldsymbol{x} \oplus_s \boldsymbol{y}) \to T(\boldsymbol{x} + \boldsymbol{y})$$

$$f_s(\boldsymbol{x}) \oplus_s f_s(\boldsymbol{y}) \to T\boldsymbol{x} + T\boldsymbol{y}$$

$$f_s(r\otimes_s\boldsymbol{x}) \to Tr\boldsymbol{x}$$

$$r\otimes_s f_s(\boldsymbol{x}) \to rT\boldsymbol{x}$$

as $s \to \infty$ が成り立つ, という性質を有する写像 f たちである. 問題によって, 収束のオーダーに関する条件を課す必要が生じるだろう.

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Bishop-Phelps-Bollobas property and property β

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1 Introduction

This is an introduction of a paper [2] by Acosta, Aron, García and Maestre and recent results. This is not my research result.

Given a Banach space X, we will use the following notation:

- $B_X := \{x \in X : \|x\| \leq 1\}$, the closed unit ball of X,
- $S_X := \{x \in X : \|x\| = 1\}$, the unit sphere of X,
- X^* : the (topological) dual space of X.

A bounded linear functional $x^* \in X^*$ is said to be *norm-attaining* if there is an $x \in S_X$ such that $|x^*(x)| = \|x^*\|$. It is well-known that every bounded linear functional on a reflexive Banach space must be norm-attaining. The converse is also true in the following sense.

If X is a Banach space and every bounded linear functional x^* on X is norm-attaining, then X is reflexive.

James proved the above result for separable Banach spaces in [8], and he generalized it to arbitrary Banach spaces in [9]

After James [8], Phelps began to study norm-attaining functionals on non-reflexive Banach spaces. The following theorem is so-called the Bishop-Phelps subreflexivity theorem that proved by Bishop and Phelps in [5].

Theorem 1. *For any Banach space X, the set of all norm-attaining members in X^* is dense in the dual space X^* of X.*

Bollobás in [6] investigated a generalization of the Bishop-Phelps subreflexivity theorem and proved so-called the Bishop-Phelps-Bollobás theorem that is stated as follows.

Theorem 2. *Let X be a Banach space and $0 < \varepsilon < 1/2$. Given $x^* \in S_{X^*}$ and $x \in S_X$ with $|x^*(x) - 1| \leq \varepsilon^2/2$, then there exist $y^* \in S_{X^*}$ and $y \in S_X$ such that $y^*(y) = 1$, $\|y - x\| < \varepsilon + \varepsilon^2$ and $\|y^* - x^*\| \leq \varepsilon$.*

2 A main result in [2]

Let us consider when a Bishop-Phelps-Bollobás type theorem holds in the vector-valued case. For real or complex Banach spaces X and Y we will denote by $L(X,Y)$ the Banach space of all bounded linear operators from X into Y.

Definition 1. Let X and Y be real or complex Banach spaces. We say that the pair (X,Y) satisfies the *Bishop-Phelps-Bollobás property for operators* if given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ satisfying the following conditions:
for all $T \in S_{L(X,Y)}$ and $x \in S_X$ with $\|T(x)\| > 1 - \delta(\varepsilon)$, there exist $S \in S_{L(X,Y)}$ and $z \in S_X$ such that $\|S(z)\| = 1$, $\|z - x\| < \varepsilon$ and $\|S - T\| < \varepsilon$.

A bounded linear operator $T : X \to Y$ will be called *norm-attaining* if there is an $x \in S_X$ such that $\|T(x)\| = \|T\|$. Note that if the pair (X,Y) satisfies the Bishop-Phelps-Bollobás property for operators, then the set of all norm-attaining operators is norm dense in $L(X,Y)$. Lindenstrauss in [10] introduced a sufficient condition of a Banach space Y in order that the set of all norm-attaining operators be dense in $L(X,Y)$ for every Banach space X, that is stated as follows.

Definition 2. A Banach space Y is said to have the *property* β if there exist two sets $\{y_\alpha : \alpha \in A\} \subset S_Y$, $\{y^*_\alpha : \alpha \in A\} \subset S_{Y^*}$ and $0 \le \rho < 1$ such that the following conditions hold:

- $y^*_\alpha(y_\alpha) = 1$, for all $\alpha \in A$,
- $|y^*_\alpha(y_\beta)| \le \rho$, for all distinct $\alpha, \beta \in A$,
- $\|y\| = \sup\{|y^*_\alpha(y)| : \alpha \in A\}$, for all $y \in Y$.

Acosta, Aron, García and Maestre in [2] proved the following theorem.

Theorem 3. *Let X and Y be Banach spaces. If Y has the property β, then the pair (X,Y) satisfies the Bishop-Phelps-Bollobás property for operators.*

Note that Partington in [11] showed that every Banach space can be equivalently renormed to have the property β.

Theorem 4. *Every Banach space has a norm $|\cdot|$ equivalent to the original norm such that $(X, |\cdot|)$ has the property β.*

As a corollary, for any Banach space Y there is a Banach space Z isomorphic to Y such that (X,Z) satisfies the Bishop-Phelps-Bollobás property for operators for every Banach space X.

Acosta, Aron, García and Maestre also proved that if X and Y are finite-dimensional Banach spaces, then the pair (X,Y) satisfies the Bishop-Phelps-Bollobás property for operators. More explicitly, the following is true.

Theorem 5. *Let X and Y be finite-dimensional Banach spaces. For any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $T \in S_{L(X,Y)}$ there is $R \in S_{L(X,Y)}$ such that the following conditions hold:*

- *$\|R - T\| < \varepsilon$, and*
- *for every $x \in S_X$ with $\|T(x)\| > 1 - \delta$ there is $x' \in S_X$ such that $\|R(x')\| = 1$ and $\|x' - x\| < \varepsilon$.*

3 Some recent results

Now let us introduce more concrete examples. Let K be a compact Hausdorff space. The set of all real-valued continuous functions on K will be denoted by $C_{\mathbb{R}}(K)$. $C_{\mathbb{R}}(K)$ becomes a Banach space with the pointwise operations and the supremum norm. The next theorem is proved by Acosta, Becerra-Guerrero, Choi, Ciesielski, Kim, Lee, Lourenço and Martín in [3].

Theorem 6. *Let K_1 and K_2 be compact Hausdorff spaces. The pair $(C_{\mathbb{R}}(K_1), C_{\mathbb{R}}(K_2))$ satisfies the Bishop-Phelps-Bollobás property for operators.*

Now let $C(K)$ be a Banach space of complex-valued continuous functions on a compact Hausdorff space K with supremum norm. Note that it is not known that whether or not $(C(K_1), C(K_2))$ satisfies the Bishop-Phelps-Bollobás property for operators for any compact Hausdorff spaces K_1 and K_2. On the other hand, Cascales, Guirao and Kadets in [7] showed the following theorem.

Theorem 7. *Let X be an Asplund space, and let A be a uniform algebra on a compact Hausdorff space K. Then the pair (X, A) satisfies the Bishop-Phelps-Bollobás property for operators.*

Let $C_0(L)$ be a Banach space of complex-valued continuous functions on a locally compact Hausdorff space L which vanish at infinity with supremum norm. It is known that the class of locally compact Hausdorff space L for which $C_0(L)$ is Asplund is very small. In fact, such a locally compact Hausdorff space must be *scattered*, that is, L contains no non-empty perfect subset (see after Corollary 2.6 in [4], and [12]).

A Banach space Y is *complex uniformly convex* if

$$\inf\{\sup\{\|x + \lambda\varepsilon y\| - 1 : \lambda \in \mathbb{C}, |\lambda| = 1\} : x, y \in S_Y\} > 0$$

for every $\varepsilon > 0$. Acosta in [1] showed the following theorem.

Theorem 8. *The pair $(C_0(L), Y)$ satisfies the Bishop-Phelps-Bollobás property for operators for any locally compact Hausdorff space L and any complex uniformly convex Banach space Y.*

4 Problems

Finally, let us introduce some problems in which I am interested.

Question. Let L, L_1, L_2 be locally compact Hausdorff spaces.

- Characterize L such that $(X, C_0(L))$ satisfies the Bishop-Phelps-Bollobás property for operators for every Banach space X.
- Give a necessary and sufficient condition of L in order that the set of norm-attaining operators from X into $C_0(L)$ be dense in $L(X, C_0(L))$ for every Banach space X.
- Does the pair $(C_0(L_1), C_0(L_2))$ satisfy the Bishop-Phelps-Bollobás property for operators?
- Does the pair (A, Y) satisfy the Bishop-Phelps-Bollobás property for operators for any function algebra on L and any complex uniformly convex Banach space Y?

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Linear isometries of S^p, an introduction of a paper by Novinger and Oberlin

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This is an introduction of some results in a paper by Novinger and Oberlin [3].

Let $\mathbb{D}$ be the open unit disc $\{z \in \mathbb{C} \ : \ |z| < 1\}$ and $\mathbb{T}$ be the unit circle $\{z \in \mathbb{C} \ : \ |z| = 1\}$. For $1 \le p < \infty$ let H^p be the class of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_p := \sup_{0 \le r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < \infty,$$

we call this norm $\| \cdot \|_p$ H^p norm. Let H^∞ be the class of all bounded analytic functions on $\mathbb{D}$ with the supremum norm. Let S^p be the class of analytic functions f on $\mathbb{D}$ such that $f' \in H^p$ for $1 \le p < \infty$. We consider the following norms on S^p:

$$\|f\|_0 = |f(0)| + \|f'\|_p \ ,$$

$$\|f\|_\Sigma = \|f\|_\infty + \|f'\|_p \ ,$$

where $\|f\|_\infty = \sup\{|f(z)| \ : \ z \in \overline{\mathbb{D}}\}$.

The linear isometries of spaces of analytic functions heve been studied since the 1960's. deLeeuw, Rudin, and Wermer [1] obtained the form of the isometries for H^∞ and H^1.

Theorem 1 (deLeeuw, Rudin and Wermer, 1960) *If T is a linear isometry of H^∞ onto H^∞, then there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and a conformal map ϕ on $\mathbb{D}$ onto itself such that*

$$Tf(z) = \lambda f(\phi(z))$$

for all $f \in H^\infty$ and $z \in \mathbb{D}$.

Theorem 2 (deLeeuw, Rudin and Wermer, 1960) *If T is a linear isometry of H^1 onto H^1, then there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and a conformal map ϕ on $\mathbb{D}$ onto itself such that*

$$Tf(z) = \lambda \phi'(z) f(\phi(z))$$

for all $f \in H^1$ and $z \in \mathbb{D}$.

Their theorems were extended to H^p for $1 \le p < \infty, \ p \neq 2$ by Forelli [2].

Theorem 3 (Forelli, 1964) *Suppose that $1 \leq p < \infty$, $p \neq 2$. If T is a linear isometry of H^p into H^p, then there is a non-constant inner function $\phi \in H^\infty$ and a function $F \in H^p$ such that*

$$Tf(z) = F(z)f(\phi(z))$$

for all $f \in H^p$ and $z \in \mathbb{D}$.

Remark. In this theorem, F and ϕ are related.

If a linear isometry $T : H^p \longrightarrow H^p$ is surjective, then T comes from only a conformal map.

Theorem 4 (Forelli, 1964) *Suppose that $1 \leq p < \infty$, $p \neq 2$. If T is a linear isometry of H^p onto H^p, then there is $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and a conformal map ϕ of $\mathbb{D}$ onto itself such that*

$$Tf(z) = \lambda(\phi'(z))^{\frac{1}{p}} f(\phi(z))$$

for all $f \in H^p$ and $z \in \mathbb{D}$.

Novinger and Oberlin [3] deterimined linear isometries on S^p with respect to $\|\cdot\|_0$ and $\|\cdot\|_\Sigma$.

Theorem 5 (Novinger and Oberlin, 1985) *If T is a linear isometry of S^p into S^p with respect to*

$$\|f\|_0 = |f(0)| + \|f'\|_p,$$

then there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and a linear isometry U of H^p into H^p such that

$$Tf(z) = \lambda \left(f(0) + \int_0^z Uf'(\zeta)d\zeta \right)$$

for all $f \in S^p$ and $z \in \mathbb{D}$.

Sketch of proof. Let $n \in \mathbb{N}$, $t \in \mathbb{R}$ and $Z^n(z) = z^n$ $(z \in \mathbb{D})$. Consider $\|T(1 + tZ^n)\|_0 = \|T1(0) + tTZ^n(0)| + \|(T1)' + t(TZ^n)'\|_p$. Then we have

$$\|(T1)' + t(TZ^n)'\|_p = \|(T1)'\|_p + |t|\|(TZ^n)'\|_p.$$

We see that $T1$ is a constant function with $|T1| = 1$. We may suppose that $T1 = 1$. Since T is a linear isometry,

$$\|1 + t(f - f(0))\|_0 = \|T1 + tT(f - f(0))\|_0.$$

Then, it follows that

$$1 + |t|(\|f'\|_p - \|(Tf)'\|_p) = |1 + t(Tf(0) - f(0))|,$$

for all $t \in \mathbb{R}$. Hence

$$Tf(0) = f(0).$$

Let $S_0^p = \{f \in S^p \ : \ f(0) = 0\}$. Let $D : S_0^p \longrightarrow H^p, \ Df = f'$, then D is a linear isometry, and let I be the inverse of D, given by

$$Ig(z) = \int_0^z g(\zeta)d\zeta \qquad (g \in H^p).$$

Since $T1 = 1$ and $Tf(0) = f(0)$, then $T(f - f(0)) = Tf - Tf(0)$ and thus

$$T(If') = I((Tf)').$$

Therefore

$$D \circ T \circ I(f') = D \circ I(Tf)' = (Tf)'.$$

Let $U = D \circ T \circ I$, then $Uf' = (Tf)'$. Hence

$$Tf(z) = f(0) + \int_0^z Uf'(\zeta)d\zeta$$

for all $z \in \mathbb{D}$. □

Using Forelli's theorems, we obtain the following corollaries.

Corollary 1 (Novinger and Oberlin, 1985) *Suppose that $1 \leq p < \infty, \ p \neq 2$. If T is a linear isometry of S^p into S^p with respect to*

$$\|f\|_0 = |f(0)| + \|f'\|_p \ ,$$

then there is a non-constant inner function ϕ and a function $F \in H^p$ such that

$$Tf(z) = \lambda \left(f(0) + \int_0^z F(\zeta)f'(\phi(\zeta))d\zeta \right)$$

for all $f \in S^p$ and $z \in \mathbb{D}$.

Corollary 2 (Novinger and Oberlin, 1985) *Suppose that $1 \leq p < \infty, \ p \neq 2$. If T is a linear isometry of S^p onto S^p with respect to*

$$\|f\|_0 = |f(0)| + \|f'\|_p \ ,$$

then there are $\lambda_1, \ \lambda_2 \in \mathbb{C}$ with $|\lambda_1| = |\lambda_2| = 1$ and a conformal map ϕ of $\mathbb{D}$ onto $\mathbb{D}$ such that

$$Tf(z) = \lambda_1 \left(f(0) + \lambda_2 \int_0^z (\phi'(\zeta))^{\frac{1}{p}} f'(\phi(\zeta))d\zeta \right)$$

for all $f \in S^p$ and $z \in \mathbb{D}$.

Theorem 6 (Novinger and Oberlin, 1985) *If T is a linear isometry of S^p into S^p with respect to*

$$\|f\|_\Sigma = \|f\|_\infty + \|f'\|_p \ ,$$

then there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and a conformal map ϕ of $\mathbb{D}$ onto $\mathbb{D}$ such that

$$Tf(z) = \lambda f(\phi(z))$$

for all $f \in S^p$ and $z \in \mathbb{D}$. If $1 < p < \infty$, ϕ is necessarly a rotation of $\mathbb{D}$.

Sketch of proof. Similarly, consider $\|T(1+wZ^n)\|_\Sigma = \|T1+wTZ^n\|_\infty + \|(T1)'+w(TZ^n)'\|_p$ $(n \in \mathbb{N},\ w \in \mathbb{C})$. We have

$$\|T1+wTZ^n\|_\infty = \|T1\|_\infty + |w|\|TZ^n\|_\infty, \tag{1}$$
$$\|(T1)'+w(TZ^n)'\|_p = \|(T1)'\|_p + |w|\|(TZ^n)'\|_p. \tag{2}$$

We see that $T1$ is a constant function of modulus one by (2) as before. So we can suppose that $T1 = 1$. We may assume that $T1 = 1$. Then, $\|1+wTZ\|_\infty = 1+|w|\|TZ\|_\infty$ by (1). We can prove that $TZ(\mathbb{T})$ contains $\{z \in \mathbb{C} \ : \ |z| = \|TZ\|_\infty\}$. We see that

$$1 = \|TZ\|_\infty = \|(TZ)'\|_1 = \|(TZ)'\|_p.$$

Thus TZ is a conformal map on $\mathbb{D}$ onto itself. And if $p > 1$, TZ is a rotation of $\mathbb{D}$. Let $T_1 : S^p \longrightarrow S^p$ be defined by

$$T_1 f = Tf \circ (TZ)^{-1} \ (f \in S^p),$$

then T_1 is an isometry of S^p. Now it follows that

$$T_1 1 = T1 \circ (TZ)^{-1} = 1,$$
$$T_1 Z = TZ \circ (TZ)^{-1} = Z.$$

Therefore we can prove that

$$T_1 Z^n = Z^n$$

for all $n \in \mathbb{N}$. Since the polynomials are dense in S^p, we have

$$T_1 f = f$$

for all $f \in S^p$. Thus from this it follows that

$$f = T_1 f = Tf \circ (TZ)^{-1}$$

so that

$$Tf = f \circ TZ.$$

Hence let $\phi = TZ$, then $Tf = f \circ \phi$. □

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Isometries on C^1-spaces of $C(X)$-valued functions

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1 Introduction

Let $(M, \|\cdot\|_M)$ and $(N, \|\cdot\|_N)$ be normed linear spaces over the complex number field $\mathbb{C}$. A mapping $S\colon M \to N$ is said to be an *isometry* if and only if

$$\|S(f) - S(g)\|_N = \|f - g\|_M \qquad (f, g \in M).$$

Let $\mathbb{F}$ be the real or complex number field. We denote by $C_{\mathbb{F}}(K)$ the Banach space of all continuous $\mathbb{F}$-valued functions on a compact Hausdorff space K with point wise operations and the supremum norm $\|f\|_K = \sup_{t \in K} |f(t)|$ for $f \in C_{\mathbb{F}}(K)$. For simplicity, we will write $C(K)$ instead of $C_{\mathbb{C}}(K)$. Banach [1] gave the characterization of surjective isometries between $C_{\mathbb{R}}(X)$ and $C_{\mathbb{R}}(Y)$ for compact *metric spaces* X and Y, where $\mathbb{R}$ is the real number field. Stone [11] generalized the result by Banach to compact Hausdorff spaces X and Y. Their results are well-known as the Banach-Stone theorem. Jerison [6] investigated surjective linear isometries between Banach spaces of vector-valued continuous maps on a compact Hausdorff space. Since then, isometries on vector-valued function spaces have been studied extensively. For example, Botelho and Jamison [2] consider E-valued C^1 function space $C^1([0,1], E)$ for a Hilbert space E. For a finite dimensional E, they characterize surjective linear isometry on $C^1([0,1], E)$ with respect to a norm $\sup_{t \in [0,1]}(\|f(t)\|_E + \|f'(t)\|_E)$. In particular, if $E = \mathbb{C}$, then the corresponding result was obtained by Cambern [3]. Rao and Roy [10] proved a quite similar result to the theorem by Cambern with another norm. The purpose of this note is to characterize surjective linear isometries on $C^1([0,1], A)$ for a uniform algebra A with an additional assumption.

2 Main result

Let A be a uniform algebra on a compact Hausdorff space X with the supremum norm $\|\cdot\|_X$, that is, A is a uniformly closed subalgebra of $C(X)$ that contains constants and separates points of X in the following sense; for each $x, y \in X$ with $x \neq y$ there exists $f \in A$ such that $f(x) \neq f(y)$. Let A^* be the dual space of A, and let A_1^* be the closed unit ball of A^*. For each $x \in X$, the functional $\delta_x \in A^*$ is defined by $\delta_x(f) = f(x)$ for $f \in A$. A functional $\eta \in A_1^*$ is an extreme point if and only if $\eta = (\xi+\zeta)/2$ for $\xi, \zeta \in A_1^*$ implies that $\xi = \zeta$. Denote by $\mathrm{Ext}(A_1^*)$ the set of all extreme points of A_1^*. We define $\mathrm{Ch}(A)$ by the set of all $x \in X$ such that $\delta_x \in \mathrm{Ext}(A_1^*)$. It is well-known that the closure of $\mathrm{Ch}(A)$ in X is the Shilov boundary for A, that is, the smallest closed boundary for A. We denote by ∂A the Shilov boundary for A.

A map $F\colon [0,1] \to A$ is said to be differentiable if there exists a map $F'\colon [0,1] \to A$ with

$$\lim_{h\to 0} \left\| \frac{F(t+h)-F(t)}{h} - F'(t) \right\|_X = 0$$

for all $t \in [0,1]$; for $t = 0, 1$, the above limit means right hand and left hand one-sided limit, respectively. We denote by $C^1([0,1], A)$ the set of all differentiable map $F\colon [0,1] \to A$ such that $F'\colon [0,1] \to A$ is continuous on $[0,1]$. There are a lot of norms on $C^1([0,1], A)$. In fact, let D be a compact connected subset of $[0,1] \times [0,1]$. Then the quantity

$$\|F\|_{\langle D\rangle} = \sup_{(t_1,t_2)\in D} (\|F(t_1)\|_X + \|F'(t_2)\|_X)$$

is a norm for $F \in C^1([0,1], A)$ provided that $p_1(D) \cup p_2(D) = [0,1]$, where p_j is the projection to the j-th coordinate of $[0,1] \times [0,1]$ for $j = 1, 2$.

Now we are ready to state the main result of this paper. We characterize surjective complex linear isometries on $(C^1([0,1], A), \|\cdot\|_{\langle D\rangle})$. Roughly speaking, such isometries are represented by weighted composition operators.

Theorem. *Let A be a uniform algebra on a compact Hausdorff space X with an additional property that $\mathrm{Ch}(A) = \partial A$. Let D be a compact connected subset of $[0,1]\times[0,1]$ satisfying $p_1(D) = p_2(D) = [0,1]$. If $S\colon C^1([0,1], A) \to C^1([0,1], A)$ is a surjective, complex linear isometry with respect to $\|\cdot\|_{\langle D\rangle}$, then there exist $\beta \in A$ with $|\beta| = 1$ on $\mathrm{Ch}(A)$, a homeomorphism $\psi_1\colon \mathrm{Ch}(A) \to \mathrm{Ch}(A)$ and a closed and open subsets X_{-1}, X_1 of $\mathrm{Ch}(A)$ with $X_{-1} \cup X_1 = \mathrm{Ch}(A)$ and $X_{-1} \cap X_1 = \emptyset$ such that*

$$S(F)(t)(x) = \beta(x)F(\varphi_1(t,x))(\psi_1(x)) \qquad (F \in C^1([0,1], A),\ t \in [0,1]),$$

where φ_1 is defined by

$$\varphi_1(t,x) = \begin{cases} t & \text{if} \quad x \in X_1 \\ 1-t & \text{if} \quad x \in X_{-1} \end{cases} \qquad (t \in [0,1]).$$

Conversely, if β, X_j $(j = \pm 1)$, ψ_1 and φ_1 satisfy the properties above, then S of the above form is a complex linear isometry on $C^1([0,1], A)$ with respect to $\|\cdot\|_{\langle D\rangle}$.

Note that Dirichlet algebras A satisfy the assumption that $\mathrm{Ch}(A) = \partial A$ in the above theorem. Here, we give outline of proof of the theorem.

Outline of proof. We first embed $(C^1([0,1], A), \|\cdot\|_{\langle D\rangle})$ into $(C(\tilde{D}), \|\cdot\|_{\tilde{D}})$, where $\tilde{D} = D \times \mathrm{Ch}(A) \times \mathrm{Ch}(A) \times \mathbb{T}$ and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. In fact, for each $F \in C^1([0,1], A)$ we define $\tilde{F}\colon \tilde{D} \to \mathbb{C}$ by

$$\tilde{F}(t_1, t_2, x_1, x_2, z) = F(t_1)(x_1) + zF'(t_2)(x_2) \qquad ((t_1, t_2, x_1, x_2, z) \in \tilde{D}).$$

By the definition, we observe that $F \mapsto \tilde{F}$ is a complex linear isometry from $(C^1([0,1],A), \|\cdot\|_{\langle D\rangle})$ into $(C(\tilde{D}), \|\cdot\|_{\tilde{D}})$. Let

$$U(F) = \tilde{F} \qquad \text{for} \qquad F \in C^1([0,1],A),$$

and we define

$$B = \{\tilde{F} \in C(\tilde{D}) : F \in C^1([0,1],A)\}.$$

Then $U\colon (C^1([0,1],A), \|\cdot\|_{\langle D\rangle}) \to (B, \|\cdot\|_{\tilde{D}})$ is a surjective, complex linear isometry.

We define $T\colon (B, \|\cdot\|_{\tilde{D}}) \to (B, \|\cdot\|_{\tilde{D}})$ by $T = USU^{-1}$.

$$\begin{array}{ccc} C^1([0,1],A) & \xrightarrow{S} & C^1([0,1],A) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ B & \xrightarrow[T]{} & B \end{array}$$

Then T is a surjective complex linear isometry on B with respect to the supremum norm $\|\cdot\|_{\tilde{D}}$ on the compact Hausdorff space $\tilde{D}$. By a quite similar arguments to [7, Lemma 1.6] and [10, Lemma 3.1], we can show that $\mathrm{Ch}(B) = \tilde{D}$. By the Arens-Kelly theorem (cf. [4, Corollary 2.3.6]), we see that $\mathrm{Ext}(B_1^*) = \{\lambda\delta_q : \lambda \in \mathbb{T},\ q \in \tilde{D}\}$. Let $T^*\colon B^* \to B^*$ be the adjoint of T. Since T is a surjective linear isometry, so is T^* with respect to the operator norm. It is easy to see that T^* preserves extreme points of B_1^*, that is, $T^*(\mathrm{Ext}(B_1^*)) = \mathrm{Ext}(B_1^*)$. This implies that for each $q \in \tilde{D}$ there exist $\alpha(q) \in \mathbb{T}$ and $\Phi(q) \in \tilde{D}$ such that $T^*(\delta_q) = \alpha(q)\delta_{\Phi(q)}$. By the definition of $\tilde{D}$, there exist maps $\varphi_1, \varphi_2\colon \tilde{D} \to [0,1]$, $\psi_1, \psi_2\colon \tilde{D} \to \mathrm{Ch}(A)$ and $w\colon \tilde{D} \to \mathbb{T}$ such that $(\varphi_1(q), \varphi_2(q)) \in D$ and $\Phi(q) = (\varphi_1(q), \varphi_2(q), \psi_1(q), \psi_2(q), w(q))$ for all $q \in \tilde{D}$. It follows that

$$T(\tilde{F})(q) = T^*(\delta_q)(\tilde{F}) = \alpha(q)\delta_{\Phi(q)}(\tilde{F}) = \alpha(q)\tilde{F}(\Phi(q))$$

for all $\tilde{F} \in B$ and $q \in \tilde{D}$. Since $T = USU^{-1}$ and $U(F) = \tilde{F}$ for $F \in C^1([0,1],A)$, we obtain $\widetilde{S(F)} = USU^{-1}(U(F)) = T(\tilde{F})$ and hence $\widetilde{S(F)}(q) = \alpha(q)\tilde{F}(\Phi(q))$. By the definition of $\tilde{\cdot}$, we have

$$S(F)(t_1)(x_1) + zS(F)'(t_2)(x_2) = \alpha(q)[F(\varphi_1(q)(\psi_1(q)) + w(q)F'(\varphi_2(q))(\psi_2(q))]$$

for all $q = (t_1, t_2, x_1, x_2, z) \in \tilde{D}$.

Now we need some calculation to show that α is independent of variables t_1, t_2, x_2 and z. Moreover, there exists $\beta \in A$ such that $\beta(x_1) = \alpha(t_1, t_2, x_1, x_2, z)$ for all $(t_1, t_2, x_1, x_2, z) \in \tilde{D}$. We can also prove that φ_1 does not depend on variables t_2, x_2 and z, and thus we may write $\varphi_1(t_1, x_1)$ instead of $\varphi_1(t_1, t_2, x_1, x_2, z)$. Then, by some calculation, we can show that

$$S(F)(t_1)(x_1) = \beta(x_1)F(\varphi_1(t_1, x_1))(\psi_1(t_1, x_1))$$

for all $F \in C^1([0,1],A)$, $t_1 \in [0,1]$ and $x_1 \in \mathrm{Ch}(A)$. We need to prove that the map $\psi_1(t,x)$ is constant with respect to a variable t. To do this, we need ideas by Hatori, Oi and Takagi [5] and Oi [9]. $\square$

Remark. *We considered the case when* $p_1(D) = p_2(D) = [0,1]$ *in the main theorem. We need to consider a general case when* $p_1(D) = [a,b] \subset [0,1]$*. In particular, if, in addition,* $a = b$ *in the case, I am not sure whether* $S(\mathbf{1})'(t)(x) = 0$ *holds for* $t \in [0,1]$ *and* $x \in \mathrm{Ch}(A)$*, where* $\mathbf{1}$ *is the constant function with the value* 1*. In fact, the equality plays an important role in proof of the main theorem.*

Acknowledgement. *Professor Hatori noticed that one can prove that the representation of a surjective complex linear isometry* S *holds not only on* $\mathrm{Ch}(A)$ *but also on the maximal ideal space* M_A *of* A*. He also mentioned that the assumption that* $\mathrm{Ch}(A) = \partial A$ *in the main theorem is unnecessary. I would like to thank Professor Hatori for his valuable comments and suggestions.*

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Derivatives of Blaschke products in Bergman spaces induced by doubling weights

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1 Introduction and main result

Let $\mathcal{H}(\mathbb{D})$ be the space of all analytic functions in the open unit disc $\mathbb{D}$ of the complex plane. A function $\omega : \mathbb{D} \to [0, \infty)$ is called a (radial) weight if it is integrable over $\mathbb{D}$ and $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. For $0 < p < \infty$ and a weight ω, the weighted Bergman space A^p_ω consists of those $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\, dA(z) < \infty,$$

where $dA(z)$ is the Lebesgue area measure on $\mathbb{D}$. If $\omega(z) = (1 - |z|)^\alpha$ for some $-1 < \alpha < \infty$, then we use the notation A^p_α for A^p_ω.

Class $\mathcal{D}$ offers us a sufficient ballpark [10, 11]: A radial weight ω belongs to $\mathcal{D}$ if and only if there exist $C = C(\omega) \geqslant 1$, $\alpha = \alpha(\omega) > 0$ and $\beta = \beta(\omega) \geqslant \alpha$ such that

$$C^{-1} \left(\frac{1-r}{1-s}\right)^\alpha \widehat{\omega}(s) \leqslant \widehat{\omega}(r) \leqslant C \left(\frac{1-r}{1-s}\right)^\beta \widehat{\omega}(s), \quad 0 \leqslant r \leqslant s < 1,$$

where

$$\widehat{\omega}(z) = \int_{|z|}^1 \omega(t)\, dt, \quad z \in \mathbb{D}.$$

Moreover, two subclasses are needed:

- A weight ω belongs to $\widehat{\mathcal{D}}_p$ for $0 < p < \infty$ if

$$\sup_{0<r<1} \frac{(1-r)^p}{\widehat{\omega}(r)} \int_0^r \frac{\omega(s)}{(1-s)^p}\, ds < \infty.$$

- A weight ω belongs to $\check{\mathcal{D}}_p$ for $0 < p < \infty$ if

$$\sup_{0<r<1} \frac{(1-r)^p}{\widehat{\omega}(r)} \int_r^1 \frac{\omega(s)}{(1-s)^p}\, ds < \infty.$$

For a given sequence $\{z_n\} \subset \mathbb{D}$ satisfying $\sum_n (1 - |z_n|) < \infty$, the Blaschke product with zeros $\{z_n\}$ is defined by

$$B(z) = \prod_n \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z}_n z}, \quad z \in \mathbb{D}.$$

Here we use the interpretation $|z_n|/z_n = 1$ for $z_n = 0$. We study conditions guaranteeing that the derivative of a Blaschke product belongs to the Bergman space A^p_ω. More precisely, we give an alternative proof for the essential content of [13, Theorem 1]. Earlier this subject has been studied, for instance, in [1, 2, 3, 5, 6, 7, 8]. See also monographs [4, 9].

Recall that a sequence $\{z_n\} \subset \mathbb{D}$ is separated if

$$\inf_{n\neq k}\left|\frac{z_n - z_k}{1-\overline{z}_n z_k}\right| > 0.$$

Now we are ready to state our main result, which is a combination of [13, Theorem 1 and Corollary 4].

Theorem 1 *Let $\frac{1}{2} < p < \infty$, $\omega \in \widehat{\mathcal{D}}_p$ and B be the Blaschke product with zeros $\{z_n\}$.*

(i) *If either $\frac{1}{2} < p \leqslant 1$ and $\omega \in \widehat{\mathcal{D}}_{2p-1}$, or $1 < p < \infty$ and $\omega \in \check{\mathcal{D}}_{p-1}$, then*

$$\|B'\|^p_{A^p_\omega} \lesssim \sum_n \frac{\widehat{\omega}(z_n)}{(1-|z_n|)^{p-1}}.$$

(ii) *If $\omega \in \mathcal{D}$ and $\{z_n\}$ is a finite union of separated sequences, then*

$$\|B'\|^p_{A^p_\omega} \gtrsim \sum_n \frac{\widehat{\omega}(z_n)}{(1-|z_n|)^{p-1}}.$$

The next section consists of the proof of Theorem 1. In particular, we concentrate on case (ii).

2 Proof of Theorem 1

Let $\frac{1}{2} < p \leqslant 1$. Calculating the logarithmic derivative of B, it is easy to deduce

$$|B'(z)| \leqslant \sum_n \frac{1-|z_n|^2}{|1-\overline{z}_n z|^2}.$$

Hence the Forelli-Rudin and some standard estimates together with the hypotheses of ω give

$$\begin{aligned}\|B'\|^p_{A^p_\omega} &\leqslant \sum_n (1-|z_n|^2)^p \int_{\mathbb{D}} \frac{\omega(z)}{|1-\overline{z}_n z|^{2p}}\,dA(z)\\ &\asymp \sum_n (1-|z_n|)^p \int_0^1 \frac{\omega(r)}{(|1-|z_n|r)^{2p-1}}\,dr \lesssim \sum_n \frac{\widehat{\omega}(z_n)}{(1-|z_n|)^{p-1}}.\end{aligned}$$

For $p \geqslant 1$, the Schwarz-Pick lemma and a similar deduction as in the case $p = 1$ yield

$$\int_{\mathbb{D}} |B'(z)|^p \omega(z)\,dA(z) \leqslant \int_{\mathbb{D}} |B'(z)| \frac{\omega(z)}{(1-|z|)^{p-1}}\,dA(z) \lesssim \sum_n \frac{\widehat{\omega}(z_n)}{(1-|z_n|)^{p-1}}.$$

Hence assertion (i) is proved.

Since $\{z_n\}$ is a finite union of separated sequences, we find a constant $\delta \in (0,1)$ such that the number of zeros in each $\Delta(z_n) = \{z \in \mathbb{D} : |z_n - z| < \delta(1-|z_n|)\}$ is uniformly bounded. Moreover,

$$|B(z)| \leqslant \frac{|z-z_n|}{|1-\overline{z}_n z|} \leqslant \frac{|z-z_n|}{1-|z_n|} < \delta, \quad z \in \Delta(z_n).$$

Hence, using the hypothesis $\omega \in \mathcal{D}$, we obtain

$$\begin{aligned}
\sum_n \frac{\widehat{\omega}(z_n)}{(1-|z_n|)^{p-1}} &\asymp \sum_n \int_{\Delta(z_n)} (1-|B(z)|)^p \, dA(z) \frac{\widehat{\omega}(z_n)}{(1-|z_n|)^{p+1}} \\
&\asymp \sum_n \int_{\Delta(z_n)} (1-|B(z)|)^p \, \frac{\widehat{\omega}(z)}{(1-|z|)^{p+1}} \, dA(z) \\
&\lesssim \int_{\mathbb{D}} \left(\frac{1-|B(z)|}{1-|z|} \right)^p \frac{\widehat{\omega}(z)}{1-|z|} \, dA(z).
\end{aligned}$$

Finally assertion (ii) follows from [12, Theorem 1] and the fact that, for $0 < p < \infty$ and $\omega \in \mathcal{D}$,

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} |f(z)|^p \frac{\widehat{\omega}(z)}{1-|z|} \, dA(z), \quad f \in \mathcal{H}(\mathbb{D}).$$

This asymptotic equation can be proved by observing that

$$\widehat{\omega}(r) \asymp \int_r^1 \frac{\widehat{\omega}(s)}{1-s} \, ds, \quad 0 \leqslant r < 1,$$

and integrating by parts [14]. This completes the proof.

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Bounded subsets of classes $M^p(X)$ of holomorphic functions

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Abstract. In this note, some characterizations of boundedness in $M^p(X)$ will be described, where $M^p(X)\,(0 < p < \infty)$ are F-algebras which consist of holomorphic functions defined by maximal functions.

2010 Mathematics Subject Classification: 30H15, 32A38, 46B04

Keywords: bounded subsets, $M^p(X)$, Privalov class, Smirnov class.

1. Introduction

Let n be a positive integer. The space of n-complex variables $z = (z_1, \ldots, z_n)$ is denoted by $\mathbb{C}^n$. The unit polydisk $\{z \in \mathbb{C}^n : |z_j| < 1, 1 \leq j \leq n\}$ is denoted by U^n and the distinguished boundary $\mathbb{T}^n$ is $\{\zeta \in \mathbb{C}^n : |\zeta_j| = 1,\ 1 \leq j \leq n\}$. The unit ball $\{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^2 < 1\}$ is denoted by B_n and $S_n = \{\zeta \in \mathbb{C}^n : \sum_{j=1}^n |\zeta_j|^2 = 1\}$ is its boundary. In this paper X denotes the unit polydisk or the unit ball for $n \geq 1$ and ∂X denotes $\mathbb{T}^n$ for $X = U^n$ or S_n for $X = B_n$. The normalized (in the sense that $\sigma(\partial X) = 1$) Lebesgue measure on ∂X is denoted by $d\sigma$.

The Hardy space on X is denoted by $H^q(X)$ $(0 < q \leq \infty)$. The Nevanlinna class $N(X)$ on X is defined as the set of all holomorphic functions f on X such that

$$\sup_{0 \leq r < 1} \int_{\partial X} \log^+ |f(r\zeta)|\, d\sigma(\zeta) < \infty$$

holds. It is known that $f \in N(X)$ has a finite nontangential limit, denoted by f^*, almost everywhere on ∂X.

The Smirnov class $N_*(X)$ is defined as the set of all $f \in N(X)$ which satisfy the equality

$$\sup_{0 \leq r < 1} \int_{\partial X} \log^+ |f(r\zeta)|\, d\sigma(\zeta) = \int_{\partial X} \log^+ |f^*(\zeta)|\, d\sigma(\zeta).$$

Define a metric

$$d_{N_*(X)}(f, g) = \int_{\partial X} \log(1 + |f^*(\zeta) - g^*(\zeta)|)\, d\sigma(\zeta)$$

for $f, g \in N_*(X)$. With the metric $d_{N_*(X)}(\cdot, \cdot)$ $N_*(X)$ is an F-algebra. Recall that an F-algebra is a topological algebra in which the topology arises from a complete metric.

The Privalov class $N^p(X)$, $1 < p < \infty$, is defined as the set of all holomorphic functions f on X such that

$$\sup_{0 \leq r < 1} \int_{\partial X} \left(\log^+ |f(r\zeta)|\right)^p d\sigma(\zeta) < \infty$$

holds. It is well-known that $N^p(X)$ is a subalgebra of $N_*(X)$, hence every $f \in N^p(X)$ has a finite nontangential limit almost everywhere on ∂X. Under the metric defined by

$$d_{N^p(X)}(f,g) = \left(\int_{\partial X} (\log(1 + |f^*(\zeta) - g^*(\zeta)|))^p \, d\sigma(\zeta) \right)^{\frac{1}{p}}$$

for $f, g \in N^p(X)$, $N^p(X)$ becomes an F-algebra (cf. [7]).

Now we define the class $M^p(X)$. For $0 < p < \infty$, the class $M^p(X)$ is defined as the set of all holomorphic functions f on X such that

$$\int_{\partial X} \left(\log^+ Mf(\zeta)\right)^p d\sigma(\zeta) < \infty,$$

where $Mf(\zeta) := \sup_{0 \le r < 1} |f(r\zeta)|$ is the maximal function. The class $M^p(X)$ with $p = 1$ in the case $n = 1$ was introduced by Kim in [4]. As for $p > 0$ and $n \ge 1$, the class was considered in [1, 5]. For $f, g \in M^p(X)$, define a metric

$$d_{M^p(X)}(f,g) = \left\{ \int_{\partial X} (\log(1 + M(f-g)(\zeta)))^p \, d\sigma(\zeta) \right\}^{\frac{\alpha_p}{p}},$$

where $\alpha_p = min(1,p)$. With this metric $M^p(X)$ is also an F-algebra (see [2]).

It is well-known that the following inclusion relations hold:

$$H^q(X) \subsetneq N^p(X) \subsetneq M^1(X) \subsetneq N_*(X) \quad (0 < q \le \infty\,,\, p > 1).$$

Moreover, it is known that $N(X) \subsetneq M^p(X) \quad (0 < p < 1)$ [8].

A subset L of a linear topological space A is said to be *bounded* if for any neighborhood U of zero in A there exists a real number α, $0 < \alpha < 1$, such that $\alpha L = \{\alpha f \,;\, f \in L\} \subset U$. Yanagihara characterized bounded subsets of $N_*(X)$ in the case $n = 1$ [9]. As for $M^p(X)$ with $p = 1$ in the case $n = 1$, Kim described some characterizations of boundedness (see [4]). For $p > 1$ and $n = 1$, these characterizations were considered by Meštrović [6]. As for $N^p(X)$ with $p > 1$ in the case $n \ge 1$, Subbotin investigated the properties of boundedness [7].

In this paper, we consider some characterizations of boundedness in $M^p(X)$ with $0 < p < \infty$ in the case $n \ge 1$.

2. The results

Theorem 2.1. ([3]) *Let $0 < p < \infty$. $L \subset M^p(X)$ is bounded if and only if*

(i) there exists a $K < \infty$ such that

$$\int_{\partial X} \left(\log^+ Mf(\zeta)\right)^p d\sigma(\zeta) < K$$

for all $f \in L$;

(ii) for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{E} \left(\log^+ Mf(\zeta)\right)^p d\sigma(\zeta) < \varepsilon, \quad \textit{for all } f \in L,$$

for any measurable set $E \subset \partial X$ with the Lebesgue measure $|E| < \delta$.

Proof. *Necessity.* Let L be a bounded subset of $M^p(X)$. We put $\beta_p = max(1,p) = p/\alpha_p$.

(i) For any $\eta > 0$, there is a number $\alpha_0 = \alpha_0(\eta)$ $(0 < \alpha_0 < 1)$ such that

$$\left(d_{M^p(X)}(\alpha f, 0)\right)^{\beta_p} = \int_{\partial X} (\log(1 + |\alpha| Mf(\zeta)))^p \, d\sigma(\zeta) < \eta^{\beta_p}$$

for all $f \in L$ and $|\alpha| \le \alpha_0$. It follows that

$$\int_{\partial X} \left(\log^+ |\alpha| Mf(\zeta)\right)^p d\sigma(\zeta) < \eta^{\beta_p}$$

for all $f \in L$ and $|\alpha| \le \alpha_0$. Since

$$\log^+ Mf \le \log^+ \alpha_0 Mf + \log \frac{1}{\alpha_0},$$

using the elementary inequality

$$(a+b)^p \le 2^p(a^p + b^p) \quad (a \ge 0\,,\, b \ge 0\,,\, p > 0), \tag{2.1}$$

we have

$$\begin{aligned}
&\int_{\partial X} \left(\log^+ Mf(\zeta)\right)^p d\sigma(\zeta) \\
&\qquad \le 2^p \left(\int_{\partial X} \left(\log^+ \alpha_0 Mf(\zeta)\right)^p d\sigma(\zeta) + \int_{\partial X} \left(\log \frac{1}{\alpha_0}\right)^p d\sigma(\zeta) \right) \\
&\qquad = 2^p \left(\eta^{\beta_p} + \left(\log \frac{1}{\alpha_0}\right)^p \right) = K = \mathit{constant}.
\end{aligned}$$

Thus (i) is satisfied.

(ii) For given $\varepsilon > 0$, we take η as $\eta < (\varepsilon/2^{p+1})^{\frac{1}{\beta_p}}$ and $\alpha_0 = \alpha_0(\eta)$ as above. Next take $\delta > 0$ such that

$$\delta \left(\log \frac{1}{\alpha_0}\right)^p < \frac{\varepsilon}{2^{p+1}}.$$

Then for each set $E \subset \partial X$ with $|E| < \delta$ and for every $f \in L$, we obtain

$$\begin{aligned}
&\int_E \left(\log^+ Mf(\zeta)\right)^p d\sigma(\zeta) \\
&\qquad \le 2^p \left(\int_E \left(\log^+ \alpha_0 Mf(\zeta)\right)^p d\sigma(\zeta) + \int_E \left(\log \frac{1}{\alpha_0}\right)^p d\sigma(\zeta) \right) \\
&\qquad \le 2^p \eta^{\beta_p} + 2^p |E| \left(\log \frac{1}{\alpha_0}\right)^p \\
&\qquad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&\qquad < \varepsilon.
\end{aligned}$$

Therefore, the condition (ii) is satisfied.

Sufficiency. Let

$$V = \{g \in M^p(X)\,;\, d_{M^p(X)}(g\,,\,0) < \eta\}$$

be a neighborhood of 0 in $M^p(X)$. Take $\varepsilon > 0$ such that

$$(\log(1+\varepsilon))^p + 2^p(\log 2)^p \varepsilon + 2^p \varepsilon < \eta^{\beta_p}.$$

Then, there is a δ $(0 < \delta < \varepsilon)$ such that (ii) is satisfied. For $f \in L$, we can find an $E_f \subset \partial X$ so that

$$|\partial X \backslash E_f| < \delta, \quad \left(\log^+ Mf(\zeta)\right)^p \le \frac{K}{\delta} \quad \text{on } E_f$$

by Chebyshev's inequality. We have

$$Mf(\zeta) \le \exp\left(\frac{K}{\delta}\right)^{\frac{1}{p}} = A(\delta) = A \quad \text{on } E_f.$$

Choose α such that $0 < \alpha < \varepsilon/A$. Then, using the inequality (2.1) and

$$\log(1+x) \le \log 2 + \log^+ x \quad (x > 0),$$

we obtain, for every $f \in L$,

$$\begin{aligned}
&\left(d_{M^p(X)}(\alpha f, 0)\right)^{\beta_p} \\
&\qquad = \int_{\partial X} (\log(1 + |\alpha| Mf(\zeta)))^p \, d\sigma(\zeta) \\
&\qquad = \int_{E_f} + \int_{\partial X \setminus E_f} \\
&\qquad \le \int_{E_f} (\log(1+\varepsilon))^p \, d\sigma(\zeta) \\
&\qquad\quad + 2^p \left(\int_{\partial X \setminus E_f} (\log 2)^p \, d\sigma(\zeta) + \int_{\partial X \setminus E_f} \left(\log^+ Mf(\zeta)\right)^p d\sigma(\zeta) \right) \\
&\qquad \le (\log(1+\varepsilon))^p + 2^p (\log 2)^p \delta + 2^p \varepsilon \\
&\qquad < \eta^{\beta_p}.
\end{aligned}$$

Therefore we get $d_{M^p(X)}(\alpha f, 0) < \eta$, which shows L is a bounded subset of $M^p(X)$.

The proof of the theorem is complete.

□

Next we show a standard example of a bounded set of $M^p(X)$. The following theorem is easily proved in the same way of [4, Theorem 4.6] and [7, p.236]; therefore, we do not prove it here.

Theorem 2.2. ([3]) *Let $0 < p < \infty$. If $f \in M^p(X)$, then $f_\rho(z) = f(\rho z)$ $(z \in X\,, 0 \le \rho < 1)$ form a bounded set in $M^p(X)$.*

Let $p > 1$ and we set $|f|_{N^p(X)} := d_{N^p(X)}(f\,, 0)$. Subbotin proved an equivalent condition that a subset $L \subset N^p(X)$ $(1 < p < \infty)$ is bounded. The following is a theorem by Subbotin:

Theorem 2.3. ([7]) *Let $p > 1$. A subset $L \subset N^p(X)$ is bounded if and only if the following two conditions are satisfied:*

(i) there exists a $K < \infty$ such that $|f|_{N^p(X)} \le K$ for all $f \in L$;

(ii) for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_E \left(\log^+ |f^*(\zeta)|\right)^p d\sigma(\zeta) < \varepsilon, \quad \textit{for all } f \in L,$$

for any measurable set $E \subset \partial X$ with the Lebesgue measure $|E| < \delta$.

As shown in [1, 7], for any $p > 1$ the class $M^p(X)$ coincides with the class $N^p(X)$ and the metrics $d_{M^p(X)}$ and $d_{N^p(X)}$ are equivalent. Therefore the topologies induced by these metrics are identical on the set $M^p(X) = N^p(X)$.

The following theorem is clear; therefore the proof may be omitted.

Theorem 2.4. ([3]) *Let $p > 1$. A subset $L \subset M^p(X)$ is bounded if and only if the following two conditions are satisfied:*

(i) there exists a $K < \infty$ such that

$$\int_{\partial X} \left(\log^+ |f^*(\zeta)|\right)^p d\sigma(\zeta) < K$$

for all $f \in L$*;*

(ii) for each $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\int_E \left(\log^+ |f^*(\zeta)|\right)^p d\sigma(\zeta) < \varepsilon, \quad \textit{for all } f \in L,$$

for any measurable set $E \subset \partial X$ *with the Lebesgue measure* $|E| < \delta$*.*

Acknowledgements. The author is partly supported by the Grant for Assist KAKEN from Kanazawa Medical University (K2017-6).

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Higher-dimensional non-amenability of Lipschitz algebras over compact metric spaces

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We report our recent results ([5], [6]) on the Hochschild cohomologies (see [2] and [3]) of algebras of Lipschitz functions over compact metric spaces. For a compact metric space (M, d), let Lip$M$ be the Banach algebra of all complex-valued Lipschitz functions $f : M \to \mathbb{C}$ with the norm

$$\|f\|_L = \|f\|_\infty + L(f)$$

where $\|f\|_\infty = \sup_{p \in M} |f(p)|$ and $L(f)$ denotes the *Lipschitz constant* $L(f)$ of f:

$$L(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} \mid x, y \in X,\ x \neq y \right\}.$$

Let $\tilde{M} = M \times M \setminus \Delta M$ and let $\beta\tilde{M}$ be the Stone-Čech compactification of $\tilde{M}$. Since $M \times M$ is also a compactification of $\tilde{M}$, there exists a continuous surjection $\pi : \beta\tilde{M} \to M \times M$ such that $\pi|\pi^{-1}(\tilde{M}) : \pi^{-1}(\tilde{M}) \to \tilde{M}$ is a homeomorphism. Let

$$\hat{M} = \pi^{-1}(\Delta M)$$

and the restriction of π to $\hat{M}$ is also denoted by $\pi : \hat{M} \to \Delta M \approx M$. The Banach space $C(\hat{M})$ of all complex-valued continuous functions on $\hat{M}$ with the sup norm admits a Banach Lip$M$-bimodule structure given by

$$(f \cdot \varphi)(\omega) = (\varphi \cdot f)(\omega) = f(\pi(\omega))\varphi(\omega), \qquad f \in \mathrm{Lip}M, \varphi \in C(\hat{M}), \omega \in \hat{M},$$

which allows us to study the continuous Hochschild cohomology $\mathrm{H}^*(\mathrm{Lip}M, C(\hat{M}))$.

The space $\hat{M}$ may be regarded as a non-metrizable analogue of the "space of directions". For $f \in \mathrm{Lip}M$, let $\Phi_f : \tilde{M} \to C$ be the map defined by

$$\Phi_f(x, y) = \frac{f(x) - f(y)}{d(x, y)}, \quad (x, y) \in \tilde{M}.$$

By the Lipschitz condition, Φ_f is a bounded continuous function on $\tilde{M}$ and hence admits the unique extension, called the de Leeuw map ([1], [12],[13])

$$\beta\Phi_f : \beta\tilde{M} \to \mathbb{C}$$

which restricts to the map to the space $\hat{M}$. Let $D : \mathrm{Lip}M \to C(\hat{M})$ be the map defined by

$$Df(\omega) = \beta\Phi_f(\omega), \quad f \in \mathrm{Lip}M.$$

Then D is a derivation, that is, D is a bounded linear operator which satisfies the Leibniz rule:

$$D(fg)(\omega) = f(\pi(\omega))Dg(\omega) + g(\pi(\omega))Df(\omega), \quad f, g, \in \mathrm{Lip}M, \omega \in \hat{M}.$$

Thus for a point $\omega \in \hat{M}$ and a Lipschitz function $f \in \mathrm{Lip}M$, $Df(\omega)$ may be viewed as the "directional derivative of f in the direction ω."

A metric space (M, d) is said to *satisfy the condition (G)* if, there exists a $\delta > 0$ such that, for each pair of points $x, y \in M$ with $d(x, y) \leq \delta$, there exists a unique isometric embedding of the interval $\gamma : [0, d(x, y)] \to M$ such that $\gamma(0) = x, \gamma(d(x, y)) = y$. Riemannian manifolds and CAT(κ) geodesic metric spaces are examples of spaces satisfying the condition (G).

Theorem 1 *Let (M, d) be a compact metric space satisfying the condition (G) .*

(1) For each $n \geq 1$, the cohomology $\mathrm{H}^n(\mathrm{Lip}M, C(\hat{M}))$ has the infinite $\mathrm{Lip}M$-rank in the sense that, for each $K \geq 1$, there exist $\mathrm{Lip}M$-linearly independent K elements in $\mathrm{H}^n(\mathrm{Lip}M, C(\hat{M}))$.

(2) Let $p \in M$ and let $\mathbb{C}$ be the complex number field with the $\mathrm{Lip}M$-module structure defined by

$$f \cdot z = f(p)z, \quad f \in \mathrm{Lip}M, \ z \in \mathbb{C}.$$

Then $\dim_{\mathbb{C}} \mathrm{H}^n(\mathrm{Lip}M, \mathbb{C}) = \infty$.

In particular the global homological dimension (see [2]) of $\mathrm{Lip}M$ is infinite, which corresponds to an old result on C^r-function algebras ($r \geq 1$) due to Pugach and Kleshchev [11], [8] and exhibits a sharp contrast to the dimension theorem of Ogneva [9], [10] for the Fréchet algebra $C^\infty(M)$. The notion of alternating cocycles due to Johnson [4] plays the crucial role in the proof. Here we should mention a long-standing open problem:

Open Problem. What is the global homological dimension of $C([0, 1])$?

A Banach algebra A is said to be *amenable* if every continuous derivation $D : A \to X^*$ to the dual X^* of an arbitrary Banach A bimodule X with the A-module structure:

$$a \cdot \xi(b) = \xi(ba), \quad \xi \cdot a(b) = \xi(ab), \quad \xi \in A^*, \ a, b \in A$$

is inner, in other words, $\mathrm{H}^1(A, X^*) = 0$. This is known to be equivalent to the condition $\mathrm{H}^n(A, X^*) = 0$ for each $n \geq 1$ and for each Banach A bimodule X. The fundamental theorem of Johnson [3] states that a locally compact topological group is amenable, that is the space $L^\infty(G)$ admits a left-invariant mean, if and only if its measure algebra $M(G)$ is amenable. The above theorem indicates a strong non-amenability of Lipschitz algebras.

The space $\hat{M}$ above, as the Stone-Cech remainder $\beta\tilde{M} \setminus \tilde{M}$, has complicated topology. For example it contains no compact connected metrizable subsets which are not singletons ([7]). This implies that there is no continuous map $\sigma : M \to \hat{M}$ such that $\pi \circ \sigma = \mathrm{id}_M$ whenever M contains a path, and the map $\pi : \hat{M} \to M$ is far from being a bundle projection. In view of this, the algebra $C(M)$ of the continuous functions on M would be more natural as a coefficient of the cohomology. In this direction we have the following result.

Theorem 2 *Let M be a compact Lipschitz manifold. Then we have*

$$\mathrm{H}^1(\mathrm{Lip}M, C(M)) = 0.$$

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ON EXAMPLE 8 OF THE PAPER OF JAROSZ AND PATHAK

OSAMU HATORI

ABSTRACT. The purpose of the paper is to confirm the situation of Example 8 of [5].

1. Introduction

Let K be a compact metric space and $0 < \alpha \leq 1$. We denote the algebra of all complex-valued continuous functions on K by $C(K)$. Let $0 < \alpha \leq 1$. For $f \in C(K)$, put

$$L_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}.$$

Then L_α is called an α-Lipschitz number of f, or just a Lipschitz number of f. When $\alpha = 1$ we omit the subscript α and write only $L(f)$. The space of all $f \in C(K)$ such that $L_\alpha(f) < \infty$ is denoted by $\mathrm{Lip}_\alpha(K)$. When $\alpha = 1$ the subscript is omitted and it is written as $\mathrm{Lip}(K)$.

When $0 < \alpha < 1$ the closed subalgebra

$$\mathrm{lip}_\alpha(K) = \{f \in \mathrm{Lip}_\alpha(K) : \lim_{x \to x_0} \frac{|f(x_0) - f(x)|}{d(x_0, x)^\alpha} = 0 \text{ for every } x_0 \in K\}$$

of $\mathrm{Lip}_\alpha(K)$ is called a *little* Lipschitz algebra. There are a variety of complete norms on $\mathrm{Lip}_\alpha(K)$ and $\mathrm{lip}_\alpha(K)$. In this paper we mainly concern to the norm $\|\cdot\|_L$ of $\mathrm{Lip}_\alpha(K)$ (resp. $\mathrm{lip}_\alpha(K)$) which is defined by

$$\|f\|_L = \|f\|_{\infty(K)} + L_\alpha(f), \quad f \in \mathrm{Lip}_\alpha(K) \text{ (resp. } \mathrm{lip}_\alpha(K)).$$

The norm $\|\cdot\|_M$ of $\mathrm{Lip}_\alpha(K)$ (resp. $\mathrm{lip}_\alpha(K)$) is defined by

$$\|f\|_M = \max\{\|f\|_\infty,\ L_\alpha(f)\}, \quad f \in \mathrm{Lip}_\alpha(K) \text{ (resp. } \mathrm{lip}_\alpha(K)).$$

Note that $\mathrm{Lip}(K)$ (resp. $\mathrm{lip}_\alpha(K)$) is a Banach space with respect to $\|\cdot\|_L$ and $\|\cdot\|_M$ respectively. The norm $\|\cdot\|_L$ is multiplicative. Hence $\mathrm{Lip}(K)$ (resp. $\mathrm{lip}_\alpha(K)$) is a unital Banach algebra with respect to the norm $\|\cdot\|_L$. The norm $\|\cdot\|_M$ fails to be submultiplicative. $\mathrm{Lip}_\alpha((K,d),E)$ is isometrically isomorphic to $\mathrm{Lip}((K,d^\alpha),E)$.

Jarosz and Pathak exhibited in [5, Example 8] that a surjective isometry on $\mathrm{Lip}(K)$ and $\mathrm{lip}_\alpha(K)$ of a compact metric space K with respect to the norm $\|\cdot\|_\infty + L_\alpha(\cdot)$ is canonical in the sense that it is a weighted composition operator. After the publication of [5] some authors expressed their suspicion about the argument there and the validity of the statement there had not been confirmed. The author of the present paper finds it difficult to follow the argument given in the Example 8. There seem to be a confusion of the status of the result and it would be appropriate to clarify the current situation. In fact we have exhibited the results which contains Example 8 (see [3, 1]). The purpose of this paper is to confirm Example 8 directly, where a proof is much simpler than that in [3, 1]. One of the main tool is a theorem of Jarosz [4, Theorem] which is revisited in the next section.

2010 *Mathematics Subject Classification.* 46B04,46J10 .

Key words and phrases. isometries, Lipschitz algebras .

2. A theorem of Jarosz : isometries preserving $\mathbf{1}$

It is a classical problem to ask when is an isometry between function spaces with constants is of the canonical form. The solution depends not only on the algebraic structures of these spaces, but also on the norms in most cases. Jarosz [4] defined natural norms and provided a result that isometries between a variety of spaces equipped with *natural norms* are of canonical forms.

Theorem 1 (Jarosz [4]). *Let X and Y be compact Hausdorff spaces, let A and B be complex linear subspaces of $C(X)$ and $C(Y)$, respectively, and let $p, q \in \mathcal{P}$. Assume A and B contain constant functions, and let $\|\cdot\|_A$, $\|\cdot\|_B$ be a p-norm and q-norm on A and B, respectively. Assume next that there is a linear isometry T from $(A, \|\cdot\|_A)$ onto $(B, \|\cdot\|_B)$ with $T\mathbf{1} = \mathbf{1}$. Then if $D(p) = D(q) = 0$, or if A and B are regular subspaces of $C(X)$ and $C(Y)$, respectively, then T is an isometry from $(A, \|\cdot\|_\infty)$ onto $(B, \|\cdot\|_\infty)$.*

As is pointed out in [2, 1] the original proof of Theorem 1 needs a revision in some part. A revised proof for algebra of Lipschitz functions is given in [2] and for a general case in [1]. By Theorem 1 we have the following two corollaries.

Corollary 2. *Let K_j be a compact metric space for $j = 1, 2$. Suppose that $T : \mathrm{Lip}(K_1) \to \mathrm{Lip}(K_2)$ is a surjective complex-linear isometry with respect to the norm $\|\cdot\|_M$. Assume $T\mathbf{1} = \mathbf{1}$. Then there exists a surjective isometry $\varphi : K_2 \to K_1$ such that*

$$Tf = f \circ \varphi, \qquad f \in \mathrm{Lip}(K_1). \tag{2.1}$$

Conversely if $T : \mathrm{Lip}(K_1) \to \mathrm{Lip}(K_2)$ is of the form as (2.1), then T is a surjective isometry with respect to both of $\|\cdot\|_M$ and $\|\cdot\|_L$ such that $T\mathbf{1} = 1$.

Without the assumption that $T\mathbf{1} = \mathbf{1}$ in Corollary 2, one may expect that T is a weighted composition operator. But it is not the case. A simple counterexample is given by Weaver[6, p.242] (see also [7]).

Corollary 3. *Let K_j be a compact metric space for $j = 1, 2$. Suppose that $T : \mathrm{Lip}(K_1) \to \mathrm{Lip}(K_2)$ is a surjective complex-linear isometry with respect to the norm $\|\cdot\|_L$. Assume $T\mathbf{1} = \mathbf{1}$. Then there exists a surjective isometry $\varphi : K_2 \to K_1$ such that*

$$Tf(x) = f \circ \varphi(x), \quad f \in \mathrm{Lip}(K_1),\ x \in K_2. \tag{2.2}$$

Conversely if $T : \mathrm{Lip}(K_1) \to \mathrm{Lip}(K_2)$ is of the form as (3.1), then T is a surjective isometry with respect to both of $\|\cdot\|_M$ and $\|\cdot\|_L$ such that $T\mathbf{1} = 1$.

3. Surjective isometries on $\mathrm{Lip}(X)$ with $\|\cdot\|_L$: Example 8 in [5].

Theorem 4. [5, Example 8] *Suppose that K_j is a compact metric space for $j = 1, 2$. The map $U : \mathrm{Lip}(K_1) \to \mathrm{Lip}(K_2)$ (resp. $U : \mathrm{lip}_\alpha(X_1) \to \mathrm{lip}_\alpha(K_2)$) is a surjective isometry with respect to the norm $\|\cdot\|_L$ if and only if there exists a complex number c with the unit modulus and a surjective isometry $\varphi : K_2 \to K_1$ such that*

$$U(f)(x) = cf(\varphi(x)), \qquad x \in K_2$$

for every $f \in \mathrm{Lip}(X_1)$ (resp. $f \in \mathrm{lip}_\alpha(X_1)$).

The most difficult part of the proof is to prove that $U(\mathbf{1})$ is a constant function with the unit modulus.

Proposition 5. *There exists a complex number c with $|c| = 1$ such that $U(\mathbf{1}) = c$ on K_2.*

Without loss of generality we may assume K_2 is not a singleton. To prove Proposition 5 we apply Lemma 7. To state Lemma 7 we first define an isometry from $\mathrm{Lip}(K_j)$ into a uniformly closed space of complex-valued continuous functions. Let $j = 1, 2$. Let $\mathfrak{M}_j$ be the Stone-Čech compactification of $\{(x, x') \in K^2 : x \neq x'\}$. For $f \in \mathrm{Lip}(X_j)$, let $D_j(f)$ be the continuous extension to $\mathfrak{M}_j$ of the function $(f(x) - f(x'))/d^\alpha(x, x')$ on $\{(x, x') \in K^2 : x \neq x'\}$. Then $D_j : \mathrm{Lip}(X_j) \to C(\mathfrak{M}_j)$ is well defined. We have $\|D_j(f)\|_\infty = L_\alpha(f)$ for every $f \in \mathrm{Lip}(X_j)$. Define a map

$$I_j : \mathrm{Lip}(X_j) \to C(K_j \times \mathfrak{M}_j \times \mathbb{T})$$

by $I_j(f)(x, m, \gamma) = f(x) + \gamma D_j(f)(m)$ for $f \in \mathrm{Lip}(X_j)$ and $(x, m, \gamma) \in K_j \times M_j \times \mathbb{T}$. Note that $\mathbb{T}$ is the unit circle in the complex plane. As D_j is a complex linear map, so is I_j. Let $S_j = K_j \times M_j \times \mathbb{T}$. For simplicity we just write I and D instead of I_j and D_j without causing any confusion. For every $f \in \mathrm{Lip}(X_j)$ the supremum norm $\|I(f)\|_\infty$ on S_j of $I(F)$ is written as

$$\begin{aligned}
\|I(f)\|_\infty &= \sup\{|f(x) + \gamma D(F)(m)| : (x, m, \gamma) \in S_j\} \\
&= \sup\{|f(x)| : x \in K_j\} \\
&\qquad + \sup\{|D(F)(m)| : m \in \mathfrak{M}_j\} \\
&= \|f\|_{\infty(K_j)} + \|D(F)\|_{\infty(\mathfrak{M}_j)}.
\end{aligned}$$

The second equality follows by an inspection that γ runs through the whole $\mathbb{T}$. It follows that

$$\|I(f)\|_\infty = \|f\|_\infty + \|D(f)\|_\infty = \|f\|_L$$

for every $f \in \mathrm{Lip}(X_j)$. We have $D(1) = 0$ and $I(\mathbf{1}) = \mathbf{1}$. Hence I is a complex-linear isometry with $I(\mathbf{1}) = \mathbf{1}$. In particular, $I(\mathrm{Lip}(X_j))$ is a complex-linear closed subspace of $C(S_j)$ which contains $\mathbf{1}$. In general $I(\mathrm{Lip}(X_j))$ needs not separate the points of S_j.

Lemma 6. *Suppose that $x_0 \in K_2$ and $\mathfrak{U}$ is an open neighborhood of x_0. Then there exists functions $f_0 \in \mathrm{Lip}(X_2)$ such that $0 \leq f_0 \leq 1 = f_0(x_0)$ on K_2 and $f_0 < 1/2$ on $K_2 \setminus \mathfrak{U}$. Furthermore there exists a point (x_0, m_0, γ_0) in the Choquet boundary for $I_2(\mathrm{Lip}(X_2))$ such that $\gamma_0 D(f_0)(m_0) = \|D(f_0)\|_\infty \neq 0$.*

Lemma 7. *Suppose that $x_0 \in K_2$ and $\mathfrak{U}$ is an open neighborhood of x_0. Let $f_0 \in \mathrm{Lip}(X_2)$ be a function such that $0 \leq f_0 \leq 1 = f_0(x_0)$ on K_2, and $f_0 < 1/2$ on $K_2 \setminus \mathfrak{U}$. Let (x_0, m_0, γ_0) be a point in the Choquet boundary for $I_2(\mathrm{Lip}(X_2))$ such that $\gamma_0 D(f_0)(m_0) = \|D(f_0)\|_\infty \neq 0$. (Such functions and a point (x_0, m_0, γ_0) exist by Lemma 6.) Then for any $0 < \theta < \pi/2$, $c_\theta = (x_0, m_0, e^{i\theta}\gamma_0)$ is also in the Choquet boundary for $I(\mathrm{Lip}(X_2))$.*

By Lemma 7 we can prove Proposition 5 in the same way as the proof of Proposition 9 in [3].

Proof of Theorem 4. By Proposition $U(\mathbf{1}) = c$, $|c| = 1$. Put $U_0 = \bar{c}U$. Then $U_0 : \mathrm{Lip}(X_1) \to \mathrm{Lip}(X_2)$ is a surjective isometry with respect to the norm $\| \cdot \|_L$ such that $U_0(\mathbf{1}) = \mathbf{1}$. Then by Corollary 3 there exists a surjective isometry $\varphi : K_2 \to K_1$ such that

$$U_0(f)(x) = f \circ \varphi(x), \quad f \in \mathrm{Lip}(K_1),\ x \in K_2. \tag{3.1}$$

Then we obtain that

$$U(f)(x) = cf \circ \varphi(x), \quad f \in \mathrm{Lip}(K_1),\ x \in K_2.$$

□

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STRENGTH FUNCTIONS: A STRANGE FUNCTION SPACE ASSOCIATED TO POSITIVE SEMIDEFINITE OPERATORS

LAJOS MOLNÁR

We begin with fixing the notation. In what follows

- H is a complex Hilbert space of dimension at least 2 and,
- $B(H)$ is the algebra of all bounded linear operators on H equipped with the operator norm $\|.\|$,
- $B(H)^+$ is the cone of all positive semidefinite elements of $B(H)$,
- $B(H)^{++}$ is the cone of all positive definite (i.e. invertible positive semidefinite) elements of $B(H)$,
- $P_1(H)$ is the collection of all rank-one orthogonal projections on H.

For any $A \in B(H)^+$, consider the function $\mu(A,.)$ on $P_1(H)$ defined by

$$\mu(A,P) = \operatorname{Tr} AP, \quad P \in P_1(H)$$

(Tr is the usual trace functional). This way we clearly obtain a transformation

$$A \longmapsto \mu(A,.)$$

from $B(H)^+$ to a collection of functions over the metric space $P_1(H)$. In fact, this transformation is an isometric affine order isomorphism from the cone $B(H)^+$ equipped with the operator norm and the usual oder (coming from the notion of positive semidefinitness) into the cone of all continuous nonnegative bounded real functions on $P_1(H)$ equipped with the supremum norm and the pointwise order.

We have another representation of the elements of $B(H)^+$ via so-called strength functions which concept was introduced by Busch and Gudder in [2] as follows: To any element $A \in B(H)^+$ we associate the nonnegative bounded real function $\lambda(A,.)$ on $P_1(H)$ defined by

$$\lambda(A,P) = \sup\{t \geq 0 : tP \leq A\}, \quad P \in P_1(H).$$

The function $\lambda(A,.)$ is called the strength function of $A \in B(H)^+$.

The original motivation for Busch and Gudder to define and study this notion came from the mathematical foundations of quantum mechanics.

The research was supported by the Hungarian Academy of Sciences and by the National Research, Development and Innovation Office NKFIH, Grant No. K115383. The author is very grateful to the organizers of the conference for their exceptionally kind hospitality.

Interestingly, that concept has later been employed in different areas and applications of operator theory.

Here we report on some recent investigations relating strength functions. In fact, in what follows we present a summary of the results published in our recent paper [9].

Let us denote by $\mathscr{S}(H)$ the collection of all strength functions corresponding to the operators in $B(H)^+$. By Theorem 1 and Corollary 1 in [2], the transformation

$$A \longmapsto \lambda(A,.)$$

is a one-to-one correspondence between the sets $B(H)^+$ and $\mathscr{S}(H)$ which preserves the order in both directions. Hence the operators in $B(H)^+$ can be faithfully represented by the elements of the class $\mathscr{S}(H)$ of nonnegative bounded real valued functions. Since on spaces of bounded scalar valued functions the most natural distance is the supremum distance, this immediately offers us the possibility to define a corresponding new metric on $B(H)^+$. In what follows we study its properties.

We mention that, due to the one-to-one correspondence between $B(H)^+$ and $\mathscr{S}(H)$, the results we will present can be viewed both as results on the positive semidefinite cone of operators equipped with the new metric and as results on the function space $\mathscr{S}(H)$ equipped with the supremum distance. How one in fact views this depends on one's taste or preference.

Before formulating the results, we emphasize that in our investigations and arguments a very useful formula concerning the explicit computation of the strength function of an operator $A \in B(H)^+$ is an essential tool. To present the formula, for any $A \in B(H)^+$, we denote by R_A the range of $A^{1/2}$. Theorem 4 in [2] tells us that for any $A \in B(H)^+$ and unit vector $x \in H$, the equality

$$\lambda(A, P_x) = \begin{cases} \|A^{-1/2}x\|^{-2} & \text{if } x \in R_A \\ 0 & \text{otherwise} \end{cases}$$

holds (P_x denotes the projection onto the subspace generated by x). Here $A^{-1/2}$ means the inverse of the operator

$$A^{1/2}|_{(\ker A^{1/2})^\perp} = A^{1/2}|_{\overline{R_A}}$$

from its range R_A onto $(\ker A^{1/2})^\perp = \overline{R_A}$. In particular, we have $\lambda(A, P_x) > 0$ if and only if $x \in R_A$.

1. On the algebraic structure of the set of all strength functions

We begin with two results on the algebraic structure of $\mathscr{S}(H)$. They show that $\mathscr{S}(H)$ is a rather strange collection of functions.

Proposition 1. *The pointwise sum of two elements of $\mathscr{S}(H)$ belongs to $\mathscr{S}(H)$ if and only if one of them is a scalar multiple of the other.*

The above result shows that the set $\mathscr{S}(H)$ of all strength functions on $P_1(H)$ is certainly not a cone under the usual operation of addition and scalar multiplication.

By a famous result of Kadison on the characterization of the existence of the supremum of two elelements in the partially ordered set of self-adjoint operators [4], we have the following observation concerning the pointwise order on $\mathscr{S}(H)$.

Proposition 2. *For any $A, B \in B(H)^+$ we have that the supremum of $\{\lambda(A,.), \lambda(B,.)\}$ exists in the partially ordered set $\mathscr{S}(H)$ if and only if they are comparable, i.e., either $\lambda(A,.) \le \lambda(B,.)$ or $\lambda(B,.) \le \lambda(A,.)$.*

After this we turn to the metric properties of the new metric on $B(H)^+$ or, equivalently, to the properties of $\mathscr{S}(H)$ equipped with the supremum distance.

2. Topological properties of the Busch-Gudder metric

We define the so-called Busch-Gudder metric on $B(H)^+$ as follows:

$$d_{BG}(A,B) = \sup\{|\lambda(A,P) - \lambda(B,P)| : P \in P_1(H)\}, \quad A, B \in B(H)^+.$$

Our first observation is the following.

Proposition 3. *The topology of the metric d_{BG} and that of the operator norm $\|.\|$ coincide on $B(H)^{++}$.*

Although the topologies of the Busch-Gudder metric and the operator norm coincide on $B(H)^{++}$, they are still not equivalent meaning that there are no positive real numbers c, C such that

$$c\|A - B\| \le d_{BG}(A,B) \le C\|A - B\|, \quad A, B \in B(H)^{++}.$$

Indeed, we can give a very simple example already in the two-dimensional case. Consider the sequence

$$A_n = \begin{bmatrix} 1 & 0 \\ 0 & 1/n^2 \end{bmatrix}, \quad n \in \mathbb{N}.$$

This is clearly a Cauchy sequence in the operator norm. But it is not a Cauchy sequence in d_{BG}. The reason is that the corresponding strength functions $\lambda(A_n,.)$ are all continuous while, as it can be computed explicitly, their pointwise limit is not so.

Moreover, we observe that in contrast to the coincidence of the topologies of the metric d_{BG} and the operator norm on $B(H)^{++}$, their behavior is very much different on the singular part of $B(H)^+$. In fact, for example, on the set of projections in $B(H)^+$, d_{BG} is just the discrete metric providing the strongest possible topology while the operator norm topology on that set is definitely weaker.

In the finite dimensional case we have the following comparison on the whole set $B(H)^+$.

Proposition 4. *Let H be finite dimensional. If (A_n) is a sequence in $B(H)^+$ which converges to $A \in B(H)^+$ in the Busch-Gudder metric, then the convergence holds also in the operator norm. Consequently, the operator norm topology is weaker than the topology what d_{BG} induces.*

It follows immediately that, in the finite dimensional case, the operator norm topology is strictly weaker than the one what d_{BG} induces.

Next we consider such important properties as the connectedness and the completeness of $B(H)^+$ in the Busch-Gudder metric.

For any subspace $M \subset H$, let $\mathscr{C}_M$ denote the set of all operators $A \in B(H)^+$ for which $R_A = M$. We need the following description of the closure of the operator set $\mathscr{C}_M$ in the Busch-Gudder metric.

Proposition 5. *First, $B(H)^{++}$ is not dense in $B(H)^+$ with respect to Busch-Gudder metric although it is so with respect to the operator norm topology. Second, for any subspace $M \subset H$ with $\mathscr{C}_M \neq \emptyset$, the closure of $\mathscr{C}_M$ in the topology of d_{BG} is $\mathscr{C}_M \cup \{0\}$.*

Now, concerning the connectedness of $B(H)^+$ in the topology of d_{BG} we obtain the following statement.

Proposition 6. *For any closed subspace $M \subset H$, we have that $\mathscr{C}_M$ and hence also its closure are connected in the topology of d_{BG}. If H is finite dimensional, then $B(H)^+$ is connected in the topology of the Busch-Gudder metric.*

We remark that for any (not necessarily closed) subspace M of H, the sets $\mathscr{C}_M$ and $\mathscr{C}_M \cup \{0\}$ are always convex. This follows from Douglas' majorization theorem [3]. However, in the topology of the Busch-Gudder metric convexity does not necessarily imply connectedness since, as one can show, the addition is not continuous.

The last result in this section concerns completeness of $B(H)^+$ in the metric d_{BG}.

Proposition 7. *Assume that $M \subset H$ is a closed subspace. Then $\mathscr{C}_M \cup \{0\}$ is complete in the Busch-Gudder metric. If H is finite dimensional, then the whole space $B(H)^+$ is complete in that metric.*

3. Isometry groups

Above we have seen several differences between the topologies on $B(H)^+$ induced by the Busch-Gudder metric on the one hand and by the distance coming from the operator norm on the other hand. However, in what follows we see that the isometry groups corresponding to those two distance measures are still the same.

The surjective isometries of $B(H)^+$ with respect to the operator norm can be described as follows: Any surjective operator norm isometry ϕ of $B(H)^+$ is of the form $\phi(A) = UAU^*$, $A \in B(H)^+$, where U is either a unitary or an antiunitary operator on H. This fact can be proved by applying results of Mankiewicz [6] on the extension of surjective isometries between convex sets of normed real-linear spaces with nonempty interiors, and of Kadison [5] on the structure of linear surjective isometries between the self-adjoint parts of C^*-algebras.

The next theorem shows that the structure of surjective isometries of $B(H)^+$ with respect to the Busch-Gudder metric is just the same.

Theorem 8. *Let $\phi : B(H)^+ \to B(H)^+$ be a surjective map. It is an isometry with respect to the Busch-Gudder metric, i.e., satisfies*

$$d_{BG}(\phi(A), \phi(B)) = d_{BG}(A, B), \quad A, B \in B(H)^+.$$

if and only if there is a unitary or antiunitary operator U on H such that ϕ is of the form

$$\phi(A) = UAU^*, \quad A \in B(H)^+.$$

The necessity part of the statement is easy, one really needs to prove only the sufficiency part. The main idea is the following. We show that if ϕ is a surjective isometry, then it is an order automorphism. It means that ϕ has the property that

$$A \leq B \Longleftrightarrow \phi(A) \leq \phi(B)$$

holds for any $A, B \in B(H)^+$. The structure of all such maps was determined in our paper [7]. Theorem 1 there says that every order automorphism of $B(H)^+$ is a conjugation by an invertible bounded linear or conjugate-linear operator T on H. The proof can easily be finished by showing that in our case T is necessarily a unitary or an antiunitary operator.

The proof of the fact that ϕ is order automorphism is performed in three steps in which we give a sort of metric characterization of the order.

In the first step we give a characterization of 0 in terms of the Busch-Gudder metric.

Claim 1. For an operator $A \in B(H)^+$ we have $A \neq 0$ if and only if there is $r > 0$ with the property that for every $0 < \epsilon < (1/2)r$ there are $T, S \in B(H)^+$ such that $d_{BG}(A, T) \leq r + \epsilon$, $d_{BG}(A, S) \leq r + \epsilon$ and $d_{BG}(T, S) \geq 2r$.

This implies that ϕ sends 0 to 0 and, in particular, we can deduce that ϕ preserves the operator norm.

The next claim provides us with a metric characterization of the rank-one elements of $B(H)^+$.

Claim 2. For any given positive number $r > 0$, the operator $A \in B(H)^+$ of norm r is of rank one if and only if there is exactly one element $B \in B(H)^+$ of norm $2r$ for which $d_{BG}(A, B) \leq r$.

Finally, we complete the metric characterization of the order as follows.

Claim 3. For any $A, B \in B(H)^+$ we have $A \leq B$ if and only if for every rank-one element $S \in B(H)^+$ with $\|S\| = 2\max\{\|A\|, \|B\|\}$ we have $d_{BG}(A, S) \geq d_{BG}(B, S)$.

After this, putting together all the information we have collected along Claims 1-3, we obtain that ϕ is an order automorphism of $B(H)^+$. By Theorem 1 in [7] (or see Theorem 2.5.1 in [8]), we therefore obtain that there is an invertible bounded linear or conjugate-linear operator T on H such that $\phi(A) = TAT^*$ holds for all $A \in B(H)^+$. Since ϕ preserves the operator norm, we can readily conclude that T is necessarily either unitary or antiunitary. This finishes the proof.

We can translate the above theorem to the language of strengths functions. First we recall that the surjective isometries of $P_1(H)$ with respect to the distance of the operator norm are exactly the unitary-antiunitary conjugations $P \mapsto UPU^*$. In fact, this is essentially the content of Wigner's celebrated theorem on the structure of quantum mechanical symmetry transformations, see e.g. Theorem 1.1 in [1]. Therefore, by the theorem above we get the following corollary which is a sort of Banach-Stone like theorem concerning the collection $\mathscr{S}(H)$ of real valued functions.

Corollary 9. *The surjective supremum distance isometries of the set of all strength functions on H are exactly the composition operators of the collection $\mathscr{S}(H)$ by surjective isometries of the domain $P_1(H)$.*

To complement this corollary we present the final result which shows that any permutation of $P_1(H)$ which induces a bijective composition operator on the collection $\mathscr{S}(H)$ of functions is necessarily an isometry. This again reflects a kind of strangeness of the set $\mathscr{S}(H)$ of functions. We have the following proposition.

Proposition 10. *Let φ be a bijection of the set $P_1(H)$. Then the map $s \longmapsto s\circ\varphi$ is a bijection of the set $\mathscr{S}(H)$ of all strength functions if and only if φ is of the form*

$$\varphi(P) = UPU^*, \quad P \in P_1(H)$$

with some unitary or antiunitary operator U on H.

The proofs of the above results with more material and further references can be found in our paper [9].

4. Open problems

Finally, we raise a few open problems that we find interesting with regard to the results presented above.

1) Description of the surjective Busch-Gudder isometries of the positive definite cone $B(H)^{++}$. In the proof of Theorem 8 we strongly used the fact that $B(H)^+$ contains all rank-one projections. We believe that to solve this problem one needs to invent a quite different new approach. (Observe that $B(H)^{++}$ is not dense in $B(H)^+$ in the metric d_{BG}.)

2) Is $B(H)^{++}$ with d_{BG} a metric space of non-positively curvature? This is an exciting problem raised at the meeting by Kazuhiro Kawamura.

3) Above we have obtained some finite dimensional results concerning the space $B(H)^+$ equipped with the Busch-Gudder metric. They tell that in the finite dimensional case the operator norm topology is weaker than the topology what d_{BG} induces, $B(H)^+$ is connected and complete with respect to the Busch-Gudder metric. The natural question is that what happens to these properties in the case where H is infinite dimensional.

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CYCLIC VECTORS IN FOCK-TYPE SPACES IN MULTI-VARIABLE CASE

HANSONG HUANG KOU HEI IZUCHI

1. Introduction

Let Ω be a domain in $\mathbb{C}^d$. We denote by $Hol(\Omega)$ the space of all holomorphic functions on Ω. Let $\mathcal{B}$ be a Banach space in $Hol(\Omega)$. Then a function f is said to be cyclic if $f\mathcal{C} \cap \mathcal{B}$ is dense in $\mathcal{B}$. In studying invariant subspaces for the multiplication operators by the coordinate functions, it is important to know which function is a cyclic vector.

1.1. **Hardy space over $\mathbb{D}$.** In the groundbreaking work of studying invariant subspaces in Banach spaces of holomorphic functions, Beurling characterized invariant subspaces and cyclic vectors in the Hardy space $H^2(\mathbb{D})$ over the open unit disk $\mathbb{D}$.

Theorem 1.1 (Beurling(1949)). (i) *Let M be a closed subspace of $H^2(\mathbb{D})$. Then M is an invariant subspace in $H^2(\mathbb{D})$ if and only if $M = \varphi \cdot H^2(\mathbb{D})$ where φ is an inner function.*
(ii) *Let $f \in H^2(\mathbb{D})$. Then f is a cyclic vector in $H^2(\mathbb{D})$ if and only if f is an outer function in $H^2(\mathbb{D})$.*

1.2. **Bergman space over $\mathbb{D}$.** Let $L^2_a(\mathbb{D})$ be the Bergman space over $\mathbb{D}$.

Definition 1.2. *For $f, g \in L^2_a(\mathbb{D})$,*
(i) *we denote $g \prec f$ if $\|gq\|_{L^2_a} \leq \|fq\|_{L^2_a}$ holds for all polynomials q,*
(ii) *a function $f \in L^2_a(\mathbb{D})$ is said to be $L^2_a(\mathbb{D})$-outer if $|g(0)| \leq |f(0)|$ whenever $g \prec f$.*

In 1996, Aleman-Richter-Sundberg showed the following:

Theorem 1.3 (Aleman-Richter-Sundberg(1996)). *Let f be a function in $L^2_a(\mathbb{D})$. Then f is cyclic in $L^2_a(\mathbb{D})$ if and only if f is $L^2_a(\mathbb{D})$-outer.*

However, a function-theoretic characterization for a $L^2_a(\mathbb{D})$-outer function is far beyond touch, though some special cases are treated. For a singular inner function $S(z)$, Shapiro and Roberts characterized which

$S(z)$ is cyclic. This shows that there are many singular inner functions $S(z)$ which are cyclic in $L^2_a(\mathbb{D})$. Also Borichev-Hedenmalm proved that the invertibity in $L^2_a(\mathbb{D})$ does not imply the cyclicity in $L^2_a(\mathbb{D})$, though it seems intuitively that the invertibity implies the cyclicity. From these facts, we see that there is a big difference about cyclic vectors between $H^2(\mathbb{D})$ and $L^2_a(\mathbb{D})$, see [Du, DS, Ga].

The results obtained by Beurling and Aleman-Richter-Sundberg were extended to general $H^p(\mathbb{D})$ and $L^p_a(\mathbb{D})$ for $p \geq 1$.

2. Fock space

Let $s > 0$ and $\alpha > 0$. For a positive integer d, write

$$z = (z_1, z_2, \cdots, z_d) \in \mathbb{C}^d,$$

and $|z|^2 = |z_1|^2 + |z_2|^2 + \cdots + |z_d|^2$. Let $dV(z)$ be the Lebesgue measure on $\mathbb{C}^d$. For $1 \leq p < \infty$, let $L^p_a(\mathbb{C}^d, s, \alpha)$ denote the space of all entire functions f on $\mathbb{C}^d$ satisfying

$$\|f\|^p_{L^p_a(\mathbb{C}^d,s,\alpha)} = \int_{\mathbb{C}^d} |f(z)|^p e^{-\alpha|z|^s}\, dV(z) < \infty,$$

Then $L^p_a(\mathbb{C}^d, s, \alpha)$ is a Banach space, called the Fock type space. When $p = 2$ and $s = 2$, $L^2_a(\mathbb{C}^d, 2, \alpha)$ is the classical Fock space. In $L^p_a(\mathbb{C}^d, s, \alpha)$, by [GZ] it is known that every multiplication operator by a coordinate function is unbounded, that is, for a function $f \in L^p_a(\mathbb{C}^d, s, \alpha)$ $f\mathcal{C}$ is not contained in $L^p_a(\mathbb{C}^d, s, \alpha)$.

2.1. Fock spaces over $\mathbb{C}$.

In one-variable case, the second author has characterized the cyclic vectors in the classical Fock space $L^2_a(\mathbb{C}, 2, \alpha)$. In [Iz1], it is shown that every non-vanishing function is cyclic, and its precise form is given. Also in [Iz2, HI], the authors have completely characterized the cyclic vectors in $L^p_a(\mathbb{C}, s, \alpha)$. From these results, we see that in the case of Banach spaces of entire functions on $\mathbb{C}$, the situation is different from the case on $\mathbb{D}$.

Theorem 2.1 ([Iz2])**.** *Suppose $p \geq 1$ and s is not an integer. Then the following three conditions are equivalent:*

(i) *f is a cyclic vector in $L^p_a(\mathbb{C}, s, \alpha)$,*
(ii) *f is non-vanishing function in $L^p_a(\mathbb{C}, s, \alpha)$,*
(iii) *$f = \exp(h)$, where h is a polynomial satisfying* $\deg(h) \leq [s]$.

Theorem 2.2 ([HI])**.** *Let s be a positive integer. Then the following three conditions are equivalent:*

(i) *f is a cyclic vector in $L^p_a(\mathbb{C}, s, \alpha)$,*
(ii) *f is non-vanishing and $f\mathcal{C} \subseteq L^p_a(\mathbb{C}^d, s, \alpha)$,*

(iii) $f = \exp(h)$, *where* $h(z) = \sum_{n=0}^{s} c_n z^n$, *with* $c_n \in \mathbb{C}$ *and* $|c_s| < \frac{\alpha}{p}$.

2.2. **Fock spaces over $\mathbb{C}^d$.** How about several variables cases? We get the complete characterization for cyclic vectors in $L_a^p(\mathbb{C}^d, s, \alpha)$.

For $s \notin \mathbb{N}$, we have the following.

Theorem 2.3. *Suppose $p \geq 1$ and $s > 0$ is not an integer. Then the following conditions are equivalent:*

(i) *f is a cyclic vector in $L_a^p(\mathbb{C}^d, s, \alpha)$,*
(ii) *f is non-vanishing function in $L_a^p(\mathbb{C}^d, s, \alpha)$,*
(iii) *$f = \exp(h)$, where h is a polynomial satisfying* $\deg(h) \leq [s]$.

In the case of $s \in \mathbb{N}$, for a polynomial $h(z) = \sum_{n=0}^{s} \sum_{|\beta|=n} c_\beta z^\beta$, we define

$$\Delta_h = \max_{|z|=1} \left| \sum_{|\beta|=s} c_\beta z^\beta \right|.$$

Then we have the following theorem.

Theorem 2.4. *Suppose $p \geq 1$ and s is a positive integer. Then the following conditions are equivalent:*

(i) *f is a cyclic vector in $L_a^p(\mathbb{C}^d, s, \alpha)$,*
(ii) *$f = \exp(h)$, where*

$$h(z) = \sum_{n=0}^{s} \sum_{|\beta|=n} c_\beta z^\beta,$$

with $c_\beta \in \mathbb{C}$ and $\Delta_h < \frac{\alpha}{p}$,

(iii) *f is non-vanishing and $f\mathcal{C} \subseteq L_a^p(\mathbb{C}^d, s, \alpha)$.*

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Weighted composition operators and n-tuple multiplicativity

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This paper includes the joint work with Takeshi Miura from Niigata University.

1 Definitions and Notations

Let X and Y be locally compact Hausdorff spaces. Let $C_0(X)$ be the Banach algebra of complex-valued continuous functions on X such that

$$^{\forall}\varepsilon > 0,\ ^{\exists}K \subset X\text{: compact such that } |f(x)| < \varepsilon\ (^{\forall}x \in X \setminus K),$$

with the supremum norm $\|f\| = \sup_{x \in X} |f(x)|$.

The subset $A \subset C_0(X)$ is called a function algebra on X if A satisfies the followings:

(1) A is an algebra of $C_0(X)$ and closed under the norm $\|f\|$

(2) A separates strongly the points of X

$\overset{\text{def}}{\iff}$ (i)$^{\forall}x \in X, ^{\exists}f \in A$ with $f(x) \neq 0$ and

(ii)$^{\forall}x, y \in X$ with $x \neq y$, $^{\exists}f \in A$ with $f(x) \neq f(y)$.

Let $\sigma_\pi(f) = \{f(x) : x \in X, |f(x)| = \|f\|\}$. The set $\sigma_\pi(f)$ is called the peripheral spectrum of f. If $\sigma_\pi(f) = \{1\}$, then the function f is called a peak function.

Let $P(x) = \{u \in C_0(X) :\ \sigma_\pi(u) = \{1\} = \{u(x)\}\}$ and $P_A(x) = P(x) \cap A$. For a point $x \in X$,

$$x \text{ is a peak point of } A \overset{\text{def}}{\iff} {}^{\exists}u \in P_A(x) \text{ such that } |u(\xi)| < 1\ (^{\forall}\xi \neq x),\ \text{and}$$

$$x \text{ is a weak peak point of } A \overset{\text{def}}{\iff} {}^{\exists}\{u_\alpha\} \subset P_A(x) \text{ such that } \{x\} = \cap_{u \in \{u_\alpha\}} u^{-1}(\{1\}).$$

If X is first countable, then all $x \in X$ are peak points of $C_0(X)$.

In this paper, A and B are function algebras on X and Y, respectively. Let X and Y equal to the sets of all weak peak points of A and B, respectively. Let T be a surjection from A onto B.

2 Molnár-type theorem

2001 年，Molnár の結果により，以下が示された．

Theorem (Molnár, 2001 [5]). Let $\mathcal{X}$ be a first countable compact Hausdorff space and $C(\mathcal{X})$ be the Banach algebra of complex-valued continuous functions on $\mathcal{X}$. If a surjection $S : C(\mathcal{X}) \to C(\mathcal{X})$ satisfies $\sigma(S(f)S(g)) = \sigma(fg)$ $({}^\forall f, g \in C(\mathcal{X}))$, where $\sigma(\cdot)$ is the spectrum, then there exist a homeomorphism $\phi : \mathcal{X} \to \mathcal{X}$ and a continuous function $\omega : \mathcal{X} \to \{1, -1\}$ such that $S(f) = \omega \cdot (f \circ \phi)$ $({}^\forall f \in C(\mathcal{X}))$.

この結果は一般化され，更に多くの対象に関する研究が行われている．特に, function algebra に関するいくつかの結果を，前節で仮定した条件の場合に限定して紹介する.（実際の論文のほとんどではここで示すものより一般的な結果が発表されている.）

Theorem (Hatori-Miura-Oka-Takagi, 2009 [1]). If $\sigma_\pi(T(f)T(g)) = \sigma_\pi(fg)$ $({}^\forall f, g \in A)$, then there exist a homeomorphism $\phi : Y \to X$ and a continuous function $\omega : Y \to \{1, -1\}$ such that $T(f) = \omega \cdot (f \circ \phi)$ $({}^\forall f \in A)$.

Theorem (Tonev, 2010 [7]). If $\sigma_\pi(T(f)T(g)) \cap \sigma_\pi(fg) \neq \emptyset$ and $\sigma_\pi(T(f)) = \sigma_\pi(f)$ $({}^\forall f, g \in A)$, then there exist a homeomorphism $\phi : Y \to X$ and a continuous function $\omega : Y \to \{1, -1\}$ such that $T(f) = \omega \cdot (f \circ \phi)$ $({}^\forall f \in A)$.

Theorem (cf. [3, 7]). If $\|T(f)T(g)\| = \|fg\|$ $({}^\forall f, g \in A)$, then there exist a homeomorphism $\phi : Y \to X$ such that $|T(f)| = |f \circ \phi|$ $({}^\forall f \in A)$.

上記の流れは，Molnár による荷重合成作用素の特徴づけに用いた情報を精査し，必要な情報だけを抽出していると言える．ここで，この荷重合成作用素を特徴付ける集合（情報量）はどこまで小さくできるのか？という自然な問題が浮かぶ．2010 年の Tonev の結果を見ると, かなり一般的な条件にまで達していると思われる．しかし，付加条件がない場合はどのようになっているのか？に関しては，反例も発表されておらず，完全な解答は得られていない．しかし，別の付加条件を与えることで一つの条件式のみから同様な結果が得られ，2016 年の関数環研究集会で下記を報告した.（単位元を持つ uniform algebra に関する同様な結果は [2] で示されている.）

Theorem (T., 2016 [6]). If $\sigma_\pi(T(f)T(g)) \cap \sigma_\pi(fg) \neq \emptyset$ $({}^\forall f, g \in A)$ and X is first countable, then there exist a homeomorphism $\phi : Y \to X$ and a continuous function $\omega : Y \to \{1, -1\}$ such that $T(f) = \omega \cdot (f \circ \phi)$ $({}^\forall f \in A)$.

これらの結果の証明には, peak function に関する性質を活用している．今回は特に下記を用いる．

Strong Bishop's lemma (peak points). Let $f \in A$. If $x \in X$ is a peak point of A and $f(x) \neq 0$, then there exists a peak function $u \in P_A(x)$ such that $\sigma_\pi(fu) = \{f(x)\}$ and $|fu(\xi)| < |f(x)|$ on $X \setminus \{x\}$.

その後，この結果を更に一般化させ，n 個の積に関する条件を持つ写像の構造について，結果が得られた．本稿ではその得られた結果と証明の概略を紹介する．

3 Main theorem and outline of the proof

Theorem 1 (T.). If $n \geq 2$ is a fixed natural number, X is first countable, and

$$\sigma_\pi\left(\prod_{k=1}^{n} T(f_k)\right) \cap \sigma_\pi\left(\prod_{k=1}^{n} f_k\right) \neq \emptyset \ (^\forall f_k \in A),$$

then there exist a homeomorphism $\phi : Y \to X$ and a continuous function $\omega : Y \to \{e^{\frac{2}{n}\pi i}, \cdots, e^{\frac{2(n-1)}{n}\pi i}, 1\}$ such that $T(f) = \omega \cdot (f \circ \phi)$ $(^\forall f \in A)$.

(Outline of the proof) If $\sigma_\pi\left(\prod_{k=1}^{n} T(f_k)\right) \cap \sigma_\pi\left(\prod_{k=1}^{n} f_k\right) \neq \emptyset$**, then** $\|\prod_{k=1}^{n} T(f_k)\| = \|\prod_{k=1}^{n} f_k\|$ **and** $\|T(f)\| = \|f\|$**. For every** $f, g \in A$**, there exist peak functions** $u \in P_A(y)$ **and** $U \in P_B(y)$ **such that** $\|fg\| = \|fgu^{n-2}\| \leq \|T(f)T(g)\|\|T(u)\|^{n-2} = \|T(f)T(g)\|$ **and, similarly,** $\|T(f)T(g)\| = \|T(f)T(g)U^{n-2}\| \leq \|fg\|$**. Therefore** $\|TfTg\| = \|fg\|$**. By Theorem (cf. [3, 7]) in Section 2, there exists a homeomorphism** $\phi : Y \to X$ **such that** $|T(f)| = |f \circ \phi|$ $(^\forall f \in A)$.

Fix $y \in Y$ **and** $f \in A$ **with** $f(\phi(y)) \neq 0$**. Then there exists a peak function** $u \in P_A(\phi(y))$ **such that** $\sigma_\pi(fu^{n-1}) = \{f(\phi(y))\}$ **and** $|fu^{n-1}(x)| < |f(\phi(y))|$ **on** $X \setminus \{\phi(y)\}$**. Because** $\sigma_\pi\left(T(f)T(u)^{n-1}\right) \cap \sigma_\pi(fu^{n-1}) \neq \emptyset$**, there exists a point** $\eta \in Y$ **such that** $T(f)(\eta)T(u)(\eta)^{n-1} = f(\phi(y))$**. Since** $|f(\phi(y))| = |T(f)(\eta)T(u)(\eta)^{n-1}| = |f(\phi(\eta))u(\phi(\eta))^{n-1}|$ **and** $|fu^{n-1}(x)| < |f(\phi(y))|$ **on** $X \setminus \{\phi(y)\}$**, we can see that** $\phi(\eta) = \phi(y)$**, that is** $\eta = y$ **and** $T(f)(y)T(u)(y)^{n-1} = f(\phi(y))$.

Fix $u_1 \in P_A(\phi(y))$ **with** $|u_1(x)| < 1$ **on** $X \setminus \{\phi(y)\}$**. Then** $T(u_1)(y)^n = u_1(\phi(y)) = 1$**. For any** $u_2 \in P_A(\phi(y))$, $T(u_2)(y)T(u_1)(y)^{n-1} = u_2(\phi(y)) = 1$**. That is** $T(u_2)(y) = T(u_1)(y)$ **for all** $u_2 \in P_A(\phi(y))$**. Therefore** $T(u)(y)$ **is unique and** $T(u)^n = 1$**. Define** $\omega(y) = T(u)(y)$**. Then** ω **is continuous and** $T(f) = \omega \cdot (f \circ \phi)$ $(^\forall f \in A)$. $\square$

Theorem 2 (Jointwork with Takeshi Miura from Niigata University). Let $A = C_0(X)$ **and** $B = C_0(Y)$**. If** $n \geq 3$ **is a fixed natural number,** X **is first countable, and**

$$\sup_{y \in Y}\left|\left(\prod_{k=1}^{n} T(f_k)\right)(y) + 1\right| = \sup_{x \in X}\left|\left(\prod_{k=1}^{n} f_k\right)(x) + 1\right| \ (^\forall f_k \in C_0(X)),$$

then there exist a homeomorphism $\phi : Y \to X$**, a continuous function** $\omega : Y \to \{e^{\frac{2}{n}\pi i}, \cdots, e^{\frac{2(n-1)}{n}\pi i}, 1\}$**, and a clopen subset** $K \subset Y$ **such that**

$$T(f)(y) = \omega(y) \times \begin{cases} f(\phi(y)) & y \in K \\ \overline{f(\phi(y))} & y \in Y \setminus K \end{cases} \ (^\forall f \in C_0(X)).$$

Theorem 2 については，単位元を持つ uniform algebra に関して，$n = 2$ **の結果が** [4] **などで既に知られている．しかし function algebra に関して** $n = 2$ **の場合は未解決である．**

(Outline of the proof) If $\sup_{y \in Y}\left|\left(\prod_{k=1}^{n} T(f_k)\right)(y) + 1\right| = \sup_{x \in X}\left|\left(\prod_{k=1}^{n} f_k\right)(x) + 1\right|$**, then** $\sup_{x \in X}\left|\left(f_n \prod_{k=1}^{n-1} f_k\right)(x) + \frac{1}{m}\right| \leqq \sqrt[n]{\|f_n\|^n + \frac{2}{m^n}}\left\|\prod_{k=1}^{n-1} T(f_k)\right\| + \frac{1}{m}$**. Letting** $m \to \infty$**, we can**

see that $\left\|f_n \prod_{k=1}^{n-1} f_k\right\| \leqq \|f_n\| \left\|\prod_{k=1}^{n-1} T(f_k)\right\|$**. There exists a peak function** $u \in P(y)$ **such that** $\|u \prod_{k=1}^{n-1} f_k\|_\infty = \left\|\prod_{k=1}^{n-1} f_k\right\|$**. Therefore** $\left\|\prod_{k=1}^{n-1} f_k\right\| \leqq \left\|\prod_{k=1}^{n-1} T(f_k)\right\|$**. Similarly, we can see that** $\left\|\prod_{k=1}^{n-1} T(f_k)\right\| \leqq \left\|\prod_{k=1}^{n-1} f_k\right\|$**, that is** $\left\|\prod_{k=1}^{n-1} f_k\right\| = \left\|\prod_{k=1}^{n-1} T(f_k)\right\|$**. Moreover,** $\|TfTg\|_\infty = \|fg\|_\infty$ **and there exists a homeomorphism** $\phi : Y \to X$ **with** $|T(f)| = |f \circ \phi|$**.**

Fix $y \in Y$ **and** $f \in C_0(X)$ **with** $f(\phi(y)) \neq 0$**. There exists a peak function** $u \in P(\phi(y))$ **such that** $\sigma_\pi(fu^{n-1}) = \{f(\phi(y))\}$ **and** $|fu^{n-1}(x)| < |f(\phi(y))|$ **on** $X \setminus \{\phi(y)\}$**. Let** $\beta = \overline{f(\phi(y))}/|T(f)(y)|$**. Then** $|\beta| = 1$ **and** $\sup_{y\in Y} |T(f)(\eta)T(\beta u)(\eta)T(u)(\eta)^{n-2}(y) + 1| = |f(\phi(y))| + 1$**. There exists a point** $\eta \in Y$ **such that** $T(f)(\eta)T(\beta u)(\eta)T(u)(\eta)^{n-2} = |f(\phi(y))|$**. Since** $|f(\phi(y))| = |T(f)(\eta)T(\beta u)(\eta)T(u)(\eta)^{n-2}| = |f(\phi(\eta))u(\phi(\eta))^{n-1}|$**, we can see that** $\eta = y$**, that is** $T(f)(y)T(\beta u)(y)T(u)(y)^{n-2} = |f(\phi(y))|$**.**

Fix $u_1 \in P(\phi(y))$ **with** $|u_1(x)| < 1$ **on** $X\setminus\{\phi(y)\}$**. Then** $T(u_1)(y)^n = T(u_2)(y)T(u_1)(y)^{n-1} = 1$ **for any** $u_2 \in P(\phi(y))$**. Therefore** $T(u)(y)$ **is unique and** $T(u)^n = 1$**. Define** $\omega(y) = T(u)(y)$**. Moreover, for every** $\beta \in \mathbb{C}$ **with** $|\beta| = 1$**,** $T(\beta u)(y)$ **is unique. Since there exixts a peak function** $h \in P(\phi(y))$ **such that** $\sup_{x\in X} |\beta h(x)^n + \gamma| \leq \max\{|\beta + \gamma|, 1\} (\gamma = \pm 1, \pm i)$**,** $T(\beta u)(y) = \beta T(u)(y)$ **or** $\overline{\beta} T(u)(y)$**. Define** $K = \cap_{f\in C_0(X)}\{y \in Y : T(if)(y) = iT(f)(y)\}$**. Then** K **is clopen.** □

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Operator theoretic differences between weighted Bergman and Dirichlet spaces

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1 Introduction

Throughout this article let $\mathbb{D}$ be the open unit disk in the complex plane and $\mathcal{H}(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. For $0 < p < \infty$ and $-1 < \alpha < \infty$, let A^p_α denote the weighted Bergman space of all functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{A^p_\alpha} = (1+\alpha)\int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^\alpha \, dA(z) < \infty,$$

where $dA(z) = dxdy/\pi$ denotes the Lebesgue area measure on $\mathbb{D}$. Then the functions

$$K_\lambda(z) = \frac{1}{(1-\bar{\lambda}z)^{2+\alpha}}$$

reproduce the point-evaluations for $\lambda \in \mathbb{D}$. For $1 < p < \infty$, $\langle \cdot, \cdot \rangle$ stands for the pairing in the duality $(A^p_\alpha)^* = A^q_\alpha$, where $q = p/(p-1)$. Then

$$\|K_\lambda\|^p_{A^p_\alpha} = (1+\alpha)\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{|1-\bar{\lambda}z|^{(2+\alpha)p}} \, dA(z) \approx \frac{1}{(1-|\lambda|^2)^{(2+\alpha)(p-1)}},$$

so that

$$\|K_\lambda\|_{A^p_\alpha} \approx \frac{1}{(1-|\lambda|^2)^{(2+\alpha)/q}}.$$

For $p = 1$,

$$\|K_\lambda\|_{A^1_\alpha} \approx \log \frac{1}{1-|\lambda|^2}.$$

For $1 \le p < \infty$, using Hölder's inequality, we have

$$|f(z)| \le C\|f\|_{A^p_\alpha} \frac{1}{(1-|z|^2)^{\frac{2+\alpha}{p}}} \tag{1.1}$$

and

$$|f'(z)| \le C\|f\|_{A^p_\alpha} \frac{1}{(1-|z|^2)^{1+\frac{2+\alpha}{p}}} \tag{1.2}$$

[1]The author is partially supported by Grant-in-Aid for Scientific Research, Japan Society for the Promotion of Science (No.15K04905).

for $f \in A^p_\alpha$ and $z \in \mathbb{D}$.

The Hardy spaces H^p can be viewed as limiting spaces of weighted Bergman spaces A^p_α as α decreases to -1.

Similarly, for $0 < p < \infty$ and $-1 < \alpha < \infty$, let $\mathcal{D}^p_\alpha$ denote the weighted Dirichlet space of all functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{\mathcal{D}^p_\alpha} = |f(0)|^p + (1+\alpha)\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^\alpha \, dA(z) < \infty.$$

The space $\mathcal{D}^2_0$ is the classical Dirichlet space and $\mathcal{D}^2_1 = H^2$. If $p < \alpha + 1$, then it is easy to see that $\mathcal{D}^p_\alpha = A^p_{\alpha-p}$. Indeed, this follows from the well known estimate

$$\int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^\alpha \, dA(z) \approx \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p+\alpha} \, dA(z).$$

(See [10, Theorem 6].) On the other hand, if $p = 2+\alpha$, then $\mathcal{D}^p_{p-2}$ is the analytic Besov space.

We here summarize the growth condition of functions in the weighted Dirichlet space $\mathcal{D}^p_\alpha$. For $p > 1$, using [16, Lemma 4.26, Proposition 4.27], we obtain that for a function $f \in \mathcal{D}^p_\alpha$,

$$\begin{aligned}
|f(z) - f(0)| &= \left| \frac{1}{1+\alpha} \int_{\mathbb{D}} \frac{f'(w)(1-|w|^2)}{\overline{w}(1-z\overline{w})^{2+\alpha}}(1-|w|^2)^\alpha \, dA(w) \right| \\
&\le C \int_{\mathbb{D}} \left| \frac{f'(w)(1-|w|^2)}{(1-z\overline{w})^{2+\alpha}} \right| (1-|w|^2)^\alpha \, dA(w) \\
&\le C\|f\|_{\mathcal{D}^p_\alpha} \left(\int_{\mathbb{D}} \frac{(1-|w|^2)^q}{|1-z\overline{w}|^{(2+\alpha)q}}(1-|w|^2)^\alpha \, dA(w) \right)^{1/q},
\end{aligned}$$

where $q = p/(p-1)$.

So by [16, Lemma 3.10], we have

(1) If $1 < p < 2+\alpha$, then

$$|f(z) - f(0)| \le \frac{C\|f\|_{\mathcal{D}^p_\alpha}}{(1-|z|^2)^{\frac{2+\alpha-p}{p}}}. \tag{1.3}$$

(2) If $p = 2+\alpha$, then

$$|f(z) - f(0)| \le C\|f\|_{\mathcal{D}^p_\alpha} \left(\log \frac{1}{1-|z|^2} \right)^{1/q}. \tag{1.4}$$

(3) If $2+\alpha < p$,

$$|f(z) - f(0)| \le C\|f\|_{\mathcal{D}^p_\alpha}. \tag{1.5}$$

So, if $2+\alpha < p$, then $\mathcal{D}^p_\alpha \subset H^\infty$.

We heere study operator theoretic differences between weighted Bergman spaces and weighted Dirichlet spaces by considering the integral operator.

For a fixed function $\varphi \in \mathcal{H}(\mathbb{D})$, we define two types of integral operators on $\mathcal{H}(\mathbb{D})$:

$$S_\varphi f(z) = \int_0^z \varphi(\zeta) f'(\zeta) \, d\zeta$$

and

$$T_\varphi f(z) = \int_0^z \varphi'(\zeta) f(\zeta)\, d\zeta.$$

The bilinear operator $(f,g) \to \displaystyle\int f\ g'$ was introduced by Calderón in harmonic analysis in the 60's [6]. After his research on commutators of singular integral operators, Pommerenke was probably the first author to consider the boundedness of the operator T_φ on the Hardy space in late 70's. A systematic study of T_φ in late 90's was initiated by Aleman and Siskakis. See surveys [1, 14] for more background and results on T_φ.

We consider the Bloch space as a range space of integral operators. Recall that the Bloch space $\mathcal{B}$ consists of all analytic functions f on $\mathbb{D}$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

Endowed with the norm

$$\|f\|_{\mathcal{B}_\beta} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|,$$

the Bloch space $\mathcal{B}$ becomes a Banach space. Let $\mathcal{B}_o$ be the little Bloch space consisting of all $f \in \mathcal{B}$ such that

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0.$$

In the sequel we will characterize boundedness and compactness of integral operators mapping weighted Bergman and Dirichlet spaces to the Bloch space.

To characterize the compactness, we need the following "weak convergence theorem", which is easily proved by the normal family argument.

Proposition 1.1 *Let $\mathcal{X} = A^p_\alpha, H^\infty$ and $\mathcal{D}^p_\alpha$ for $1 \le p < \infty$ and $\alpha > -1$. Suppose that S_φ $(T_\varphi, resp.) : \mathcal{X} \to \mathcal{B}$ is bounded. Then S_φ $(T_\varphi, resp.) : \mathcal{X} \to \mathcal{B}$ is compact if and only if for any bounded sequence $\{f_n\}$ in $\mathcal{X}$ that converges to 0 uniformly on every compact subset of $\mathbb{D}$, $\|S_\varphi f_n\|_{\mathcal{B}}$ $(\|T_\varphi f_n\|_{\mathcal{B}}, resp.)$ converges to 0.*

2 $A^p_\alpha \to \mathcal{B}$

Theorem 2.1 *For $1 \le p < \infty$ and $\alpha > -1$, $S_\varphi : A^p_\alpha \to \mathcal{B}$ is bounded if and only if $\varphi \equiv 0$. Moreover this equivalence also holds for any Hardy space H^p with $1 \le p < \infty$.*

Theorem 2.2 *For $1 \le p < \infty$ and $\alpha > -1$, $T_\varphi : A^p_\alpha \to \mathcal{B}$ is bounded if and only if*

(i) If $p < 2+\alpha$, then $\varphi \equiv$ constant.

(ii) If $p \ge 2+\alpha$, then $\sup_{z\in\mathbb{D}}(1-|z|^2)^{1-(2+\alpha)/p}|\varphi'(z)| < \infty$.

Theorem 2.3 *For $1 \leq p < \infty$, $T_\varphi : H^p \to \mathcal{B}$ is bounded if and only if*

(i) *If $p = 1$, then $\varphi' \in H^\infty$.*

(ii) *If $1 < p < \infty$, then $\sup_{z \in \mathbb{D}} (1 - |z|^2)^{1-1/p} |\varphi'(z)| < \infty$.*

Next we consider the compactness of T_φ.

Theorem 2.4 *Suppose that $T_\varphi : A^p_\alpha \to \mathcal{B}$ is bounded for $p \geq 2+\alpha$. Then $T_\varphi : A^p_\alpha \to \mathcal{B}$ is compact if and only if*

$$\lim_{|z| \to 1} (1 - |z|^2)^{1-(2+\alpha)/p} |\varphi'(z)| = 0.$$

Theorem 2.5 *For $1 \leq p < \infty$, suppose that $T_\varphi : H^p \to \mathcal{B}$ is bounded. Then $T_\varphi : H^p \to \mathcal{B}$ is compact if and only if*

(i) *If $p = 1$, then $\varphi \equiv$ constant.*

(ii) *If $1 < p < \infty$, then $\lim_{|z| \to 1} (1 - |z|^2)^{1-1/p} |\varphi'(z)| = 0$.*

3 $H^\infty \to \mathcal{B}$

The opertaor S_φ was not considered in [7].

Theorem 3.1 *$S_\varphi : H^\infty \to \mathcal{B}$ is bounded if and only if $\varphi \in H^\infty$.*

Theorem 3.2 *Suppose $S_\varphi : H^\infty \to \mathcal{B}$ is bounded. Then $S_\varphi : H^\infty \to \mathcal{B}$ is compact if and only if $\varphi \equiv 0$.*

Theorem 3.3 *$T_\varphi : H^\infty \to \mathcal{B}$ is bounded if and only if $\varphi \in \mathcal{B}$.*

More we can prove the following by the same as in the proof of Theorem 3.2.

Theorem 3.4 *Suppose that $T_\varphi : H^\infty \to \mathcal{B}$ is bounded. Then $T_\varphi : H^\infty \to \mathcal{B}$ is compact if and only if $\varphi \in \mathcal{B}_o$.*

Remark. Take $\varphi(z) = \log \dfrac{1}{1-z}$. Then $\varphi \in \mathcal{B}$ and T_φ is a Cesàro operator. By Theorem 3.3, $T_\varphi : H^\infty \to \mathcal{B}$ is bounded. In [8], it is shown that the Cesàro operator is bounded from H^∞ to BMOA.

4 $\mathcal{D}^p_\alpha \to \mathcal{B}$

Theorem 4.1 *For $1 \le p < 2+\alpha$ and $\alpha > -1$, $S_\varphi : \mathcal{D}^p_\alpha \to \mathcal{B}$ is bounded if and only if $\varphi \equiv 0$.*

Theorem 4.2 *For $2+\alpha < p$ and $\alpha > -1$, $S_\varphi : \mathcal{D}^p_\alpha \to \mathcal{B}$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{1-\frac{2+\alpha}{p}}|\varphi(z)| < \infty.$$

Next we consider the case of T_φ.

When $p < 1+\alpha$, the following yields from that $\mathcal{D}^p_\alpha = A^p_{\alpha-p}$.

Theorem 4.3 *For $1 < p < 1+\alpha$, $T_\varphi : \mathcal{D}^p_\alpha \to \mathcal{B}$ is bounded if and only if*

(i) if $p < \dfrac{2+\alpha}{2}$,then $\varphi \equiv$ constant.

(ii) if $\dfrac{2+\alpha}{2} \le p < 1+\alpha$,then $\sup_{z\in\mathbb{D}}(1-|z|^2)^{2-(2+\alpha)/p}|\varphi'(z)| < \infty$.

Theorem 4.4 *For $1+\alpha \le p < 2+\alpha$, $T_\varphi : \mathcal{D}^p_\alpha \to \mathcal{B}$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{2-(2+\alpha)/p}|\varphi'(z)| < \infty.$$

Theorem 4.5 *For $p = 2+\alpha$, $T_\varphi : \mathcal{D}^p_\alpha \to \mathcal{B}$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)^{1/q}|\varphi'(z)| < \infty,$$

where $q = p/(p-1)$.

If $2+\alpha < p$ and $\alpha > -1$, the following holds from the fact that $\mathcal{D}^p_\alpha \subset H^\infty$.

Theorem 4.6 *For $2+\alpha < p$ and $\alpha > -1$, $T_\varphi : \mathcal{D}^p_\alpha \to \mathcal{B}$ is bounded if and only if $\varphi \in \mathcal{B}$, that is, $\sup_{z\in\mathbb{D}}(1-|z|^2)|\varphi'(z)| < \infty$*

Next we consider the compactness of S_φ and T_φ and would obtain the "little-oh" condition in the sequel.

For the operator S_φ, it is sufficient to consider the cases $p = 2+\alpha$ and $2+\alpha < p$.

Theorem 4.7 *For $p = 2+\alpha$ and $\alpha > -1$, supose that $S_\varphi : \mathcal{D}^p_\alpha \to \mathcal{B}$ is bounded. Then $S_\varphi : \mathcal{D}^p_\alpha \to \mathcal{B}$ is compact if and only if $\varphi \equiv 0$.*

Theorem 4.8 *For $2+\alpha < p$ and $\alpha > -1$, suppose that $S_\varphi : \mathcal{D}^p_\alpha \to \mathcal{B}$ is bounded. Then $S_\varphi : \mathcal{D}^p_\alpha \to \mathcal{B}$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^{1-\frac{2+\alpha}{p}}|\varphi'(z)| = 0.$$

Next we consider the case of T_φ.

When $p < 1+\alpha$, the following yields from that $\mathcal{D}^p_\alpha = A^p_{\alpha-p}$.

Theorem 4.9 *For $p < 1+\alpha$ and $\alpha > -1$, suppose that $T_\varphi : \mathcal{D}^p_\alpha \to \mathcal{B}$ is bounded. Then the following hold.*

(i) If $p < \dfrac{2+\alpha}{2}$, then T_φ is always compact.

(ii) If $\dfrac{2+\alpha}{2} \le p < 1+\alpha$, then $T_\varphi : \mathcal{D}^p_\alpha \to \mathcal{B}$ is compact if and only if

$$\lim_{|z|\to 1}(1-|z|^2)^{2-(2+\alpha)/p}|\varphi'(z)| = 0.$$

.

Theorem 4.10 *For $1+\alpha \le p < 2+\alpha$ and $\alpha > -1$, suppose that $T_\varphi : \mathcal{D}^p_\alpha \to \mathcal{B}$ is bounded. Then $T_\varphi : \mathcal{D}^p_\alpha \to \mathcal{B}$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^{2-(2+\alpha)/p}|\varphi'(z)| = 0.$$

Theorem 4.11 *For $p = 2+\alpha$, suppose that $T_\varphi : \mathcal{D}^p_\alpha \to \mathcal{B}$ is bounded. Then $T_\varphi : \mathcal{D}^p_\alpha \to \mathcal{B}$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)^{1/q}|\varphi'(z)| = 0,$$

where $q = p/(p-1)$.

The case that $2+\alpha < p$ and $\alpha > -1$ remains.

Problem. For $2+\alpha < p$ and $\alpha > -1$, suppsoe that $T_\varphi : \mathcal{D}^p_\alpha \to \mathcal{B}$ is bounded. Then $T_\varphi : \mathcal{D}^p_\alpha \to \mathcal{B}$ is compact if and only if $\varphi \in \mathcal{B}_o$, that is, $\lim_{|z|\to 1}(1-|z|^2)|\varphi'(z)| = 0$.

The author would like to thank Dr. Atte Reijonen for pointing out a paper [12] in the conference and so has known that only inner functions in $\mathcal{D}^p_{p-2}$ are finite Blaschke products. For $2+\alpha < p$ and $\alpha > -1$, then $\mathcal{D}^p_\alpha \subset \mathcal{D}^p_{p-2}$.

Also, the author thanks Professor Takuya Hosokawa for introducing a paper [15] after the conference.

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Finite Rudin type invariant subspaces

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Let H be a separable Hilbert space and T be a bounded linear operator on H. Let M be a closed subspace of H. M is called an invariant subspace for T if $TM \subset M$. For a subset X of H, we denote by $[X]_T$ the smallest invariant subspace of H containing X for T. A subset X of M is said to be a generating set of M for T if $[X]_T = M$. Note that there are many generating sets for an invariant subspace M. The minimum number of elements in the generating sets of M is called the rank of M for T, and we denote it by $\mathrm{rank}_T M$.

Let $H^2 = H^2(\mathbb{D}^2)$ be the Hardy space over the bidisk $\mathbb{D}^2$ with variables z, w. We denote by $H^2(z), H^2(w)$ the z, w variable Hardy spaces on $\mathbb{D}$, respectively. Then H^2 coincides with the Hilbert space tensor product $H^2(z) \otimes H^2(w)$. We denote by T_z, T_w the multiplication operators on H^2 multiplying z, w, respectively. A closed subspace M of H^2 is said to be invariant if $T_z M \subset M$ and $T_w M \subset M$. The structure of invariant subspaces of H^2 is fairly complicated, and there are a lot of studies on them (see [1, 2, 7, 10, 11]). We denote by $\mathbb{T}$ the unit circle. Let $L^2 = L^2(\mathbb{T}^2)$ be the space of square integrable functions on $\mathbb{T}^2$ with respect to the normalized Lebesgue measure. Identifying $f \in H^2$ with its boundary function f^* on $\mathbb{T}^2$, we may think of H^2 as a closed subspace of L^2. For a closed subspace E of H^2, we denote by P_E the orthogonal projection from L^2 onto E.

A function $f(z) \in H^2(z)$ is called outer if $[f(z)]_{T_z} = H^2(z)$. Also $f(z)$ is called inner if $|f^*| = 1$ a.e. on $\mathbb{T}$. We denote by $\mathcal{I}$ the set of non-constant inner functions. For $\theta_1(z), \theta_2(z) \in \mathcal{I}$, we write $\theta_1(z) \prec \theta_2(z)$ if $\theta_2(z)/\theta_1(z) \in H^2(z)$. We denote by $H^\infty(z)$ the set of bounded analytic functions on $\mathbb{D}$. For $\eta \in H^\infty(z)$, we define the Toeplitz operator T_η on $H^2(z)$ by $T_\eta f = \eta f$ for $f \in H^2(z)$. For $\theta(z) \in \mathcal{I}$, write

$$K_\theta(z) = H^2(z) \ominus \theta(z) H^2(z),$$

and we denote by $S_{\eta,\theta}$ the compression operator of T_η on $K_\theta(z)$ defined by $S_{\eta,\theta} f = P_{K_\theta(z)} \eta f$ for $f \in K_\theta(z)$.

Let $\{\varphi_n(z)\}_{n \geq 1}$ be a sequence in $\mathcal{I}$ satisfying that $\varphi_{n+1}(z) \prec \varphi_n(z)$ for every $n \geq 1$. Let

$$M_1 = \bigvee_{n=1}^{\infty} \varphi_n(z) w^{n-1} H^2$$

be the closed linear span of $\varphi_n(z) w^{n-1} H^2, n \geq 1$. Then M_1 becomes an invariant subspace of H^2. In [7, p. 72], Rudin showed the existence of $\{\varphi_n(z)\}_{n \geq 1}$ satisfying that $\mathrm{rank}_{\{T_z, T_w\}} M_1 = \infty$, so M_1 is called a Rudin type invariant subspace. See [3, 4, 5, 6, 8, 9] for the studies of related subjects.

Associated with an invariant subspace M of H^2, Yang [10, 11] introduced the fringe operator $\mathcal{F}_{z,M}$ on $M \ominus wM$ defined by

$$\mathcal{F}_{z,M} = P_{M\ominus wM} T_z|_{M\ominus wM}.$$

Since

$$M = \bigoplus_{n=0}^{\infty} w^n (M \ominus wM),$$

a lot of information of M are encoded in the properties of the fringe operator. If E is a generating set of M as an invariant subspace, then $P_{M\ominus wM}E$ is a generating set of $M \ominus wM$ for $\mathcal{F}_{z,M}$. So we have

$$\operatorname{rank}_{\mathcal{F}_{z,M}}(M \ominus wM) \leq \operatorname{rank}_{\{T_z,T_w\}} M.$$

Hence to study $\operatorname{rank}_{\{T_z,T_w\}} M$, first we need to study $\operatorname{rank}_{\mathcal{F}_{z,M}}(M \ominus wM)$, for, the space $M \ominus wM$ is much smaller than M.

We have

$$M_1 \ominus wM_1 = \varphi_1(z)H^2(z) \oplus \bigoplus_{n=2}^{\infty} \varphi_n(z) w^{n-1} K_{\varphi_{n-1}/\varphi_n}(z).$$

In [4] (see also [5]), $\operatorname{rank}_{\mathcal{F}_{z,M_1}}(M_1 \ominus wM_1)$ was determined. When $\varphi_1(z)$ is a Blaschke product, moreover it is proved that

$$\operatorname{rank}_{\{T_z,T_w\}} M_1 = \operatorname{rank}_{\mathcal{F}_{z,M_1}}(M_1 \ominus wM_1).$$

Let k be a (fixed) positive integer. Let $\varphi_1(z), \varphi_2(z), \cdots, \varphi_k(z)$ and $\psi_1(w), \psi_2(w), \cdots, \psi_k(w)$ be non-constant one variable inner functions satisfying that

$$(\#1) \qquad \varphi_k(z) \prec \varphi_{k-1}(z) \prec \cdots \prec \varphi_1(z)$$

and

$$(\#2) \qquad \psi_1(w) \prec \psi_2(w) \prec \cdots \prec \psi_k w).$$

We put $\varphi_{k+1}(z) = \psi_0(w) = 1$. Let

$$\mathcal{M} = \bigvee_{n=0}^{k} \varphi_{n+1}(z_1)\psi_n(z_2)H^2.$$

By conditions (#1) and (#2), $\mathcal{M}$ is an invariant subspace of H^2 and $\mathcal{M} \neq H^2$. The space $\mathcal{M}$ is called a finite Rudin type invariant subspace and studied in [6]. For $1 \leq n \leq k$, let

$$\zeta_n(z) = \frac{\varphi_n(z)}{\varphi_{n+1}(z)} \quad \text{and} \quad \xi_n(w) = \frac{\psi_n(w)}{\psi_{n-1}(w)}.$$

By condition (#1) and (#2), $\zeta_n(z)$ and $\xi_n(w)$ are inner functions. Note that $\zeta_k(z) = \varphi_k(z)$, $\xi_1(w) = \psi_1(w)$, and

$$\varphi_\ell(z) = \prod_{n=\ell}^{k} \zeta_n(z) \quad \text{and} \quad \psi_\ell(w) = \prod_{n=1}^{\ell} \xi_n(w)$$

for every $1 \leq \ell \leq k$. Moreover, we assume that

$$(\#3) \qquad \zeta_1(z), \cdots, \zeta_k(z), \xi_1(w), \cdots, \xi_k(w) \text{ are non-constant.}$$

Note that

$$\mathcal{M} = \varphi_1(z)H^2(z) \oplus \bigoplus_{n=2}^{k+1} \varphi_n(z)\psi_{n-1}(w)K_{\zeta_{n-1}}(z) \otimes H^2(w).$$

We have

$$\mathcal{M} \ominus w\mathcal{M} = \varphi_1(z)H^2(z) \oplus \bigoplus_{n=2}^{k+1} \varphi_n(z)\psi_{n-1}(w)K_{\zeta_{n-1}}(z).$$

The purpose of this talk is to determine

$$\mathrm{rank}_{\mathcal{F}_{z,\mathcal{M}}}(\mathcal{M} \ominus w\mathcal{M}) \text{ and } \mathrm{rank}_{\mathcal{F}^*_{z,\mathcal{M}}}(\mathcal{M} \ominus w\mathcal{M}).$$

For a closed subspace E of $\mathcal{M} \ominus w\mathcal{M}$, we denote by $\mathcal{F}_{z,E}$ the compression operator of $\mathcal{F}_{z,\mathcal{M}}$ on E defined by $\mathcal{F}_{z,E}f = P_E\mathcal{F}_{z,\mathcal{M}}f$ for $f \in E$. Let $\varphi_0(z)$ be another non-constant inner function satisfying that $\varphi_1(z) \prec \varphi_0(z)$. Put $\zeta_0(z) = \varphi_0(z)/\varphi_1(z)$. Set

$$\Gamma = \bigoplus_{n=1}^{k+1} \varphi_n(z)\psi_{n-1}(w)K_{\zeta_{n-1}}(z).$$

Then

$$\mathcal{M} \ominus w\mathcal{M} = \Gamma \oplus \varphi_0(z)H^2(z) \quad \text{and} \quad \mathrm{rank}_{\mathcal{F}_{z,\Gamma}}\Gamma \leq \mathrm{rank}_{\mathcal{F}_{z,\mathcal{M}}}(\mathcal{M} \ominus w\mathcal{M}).$$

We shall determine $\mathrm{rank}_{\mathcal{F}_{z,\Gamma}}\Gamma$, $\mathrm{rank}_{\mathcal{F}^*_{z,\Gamma}}\Gamma$,

$$\mathrm{rank}_{\mathcal{F}_{z,\mathcal{M}}}(\mathcal{M} \ominus w\mathcal{M}) \text{ and } \mathrm{rank}_{\mathcal{F}^*_{z,\mathcal{M}}}(\mathcal{M} \ominus w\mathcal{M}),$$

and prove that

$$\mathrm{rank}_{\mathcal{F}_{\lambda,\Gamma}}\Gamma \leq \mathrm{rank}_{\mathcal{F}_{\lambda,\mathcal{M}}}(\mathcal{M} \ominus b_{\lambda_2}(z_2)\mathcal{M}) \leq \mathrm{rank}_{\mathcal{F}_{\lambda,\Gamma}}\Gamma + 1.$$

We may study

$$\mathrm{rank}_{\mathcal{F}_{z,\mathcal{M}_1}}(\mathcal{M}_1 \ominus w\mathcal{M}_1) \text{ and } \mathrm{rank}_{\mathcal{F}^*_{z,\mathcal{M}_1}}(\mathcal{M}_1 \ominus w\mathcal{M}_1),$$

for an infinite Rudin type invariant subspace $\mathcal{M}_1$ under some additional condition.

For

$$\begin{aligned} G &= \varphi_1(z)h_1(z) \oplus \bigoplus_{n=2}^{k+1} \varphi_n(z)\psi_{n-1}(w)h_{n-1}(z) \\ &\in \varphi_1(z)H^2(z) \oplus \bigoplus_{n=2}^{k+1} \varphi_n(z)\psi_{n-1}(w)K_{\zeta_{n-1}}(z) = \mathcal{M} \ominus w\mathcal{M}, \end{aligned}$$

we define

$$\begin{aligned} \Phi G &= \varphi_1(z)h_1(z) \oplus \bigoplus_{n=2}^{k+1} \psi_{n-1}(0)\varphi_n(z)h_{n-1}(z) \\ &\in \varphi_1(z)H^2(z) \oplus \bigoplus_{n=2}^{k+1} \varphi_n(z)K_{\zeta_{n-1}}(z) = H^2(z). \end{aligned}$$

Then if $\psi_k(0) \neq 0$, then $\Phi : \mathcal{M} \ominus w\mathcal{M} \to H^2(z)$ is a one-to-one and onto operator. The following is a key theorem.

Theorem 1 *If $\psi_k(0) \neq 0$, then $\Phi\mathcal{F}_{z,\mathcal{M}} = T_z\Phi$ on $\mathcal{M} \ominus w\mathcal{M}$ and $\mathrm{rank}_{\mathcal{F}_{z,\mathcal{M}}}(\mathcal{M} \ominus w\mathcal{M}) = 1$.*

Applying Theorem, when $\psi_k(0) = 0$ we may determine $\mathrm{rank}_{\mathcal{F}_{z,\mathcal{M}}}(\mathcal{M} \ominus w\mathcal{M})$. For sequence of inner functions satisfying the similar conditions as $(\#1), (\#2)$ and $(\#3)$, we may generalize our results.

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