2016年度 関数環研究集会

報　告　集

2017年 4月

2016 年度の関数環研究集会は茨城大学工学部「小平記念会館」を会場に，2016 年 12 月 2 日（金）～4 日（日）に開催されました．今回の研究集会は，韓国からの参加者もあり，16 件の発表が行われました。また，近年は 2 日間での研究集会開催が多かったのですが，今年は講演申込が多数あったこともあり，3 日間での開催としました。これまでよりもじっくりと皆様のお話を伺うことが出来たように思いますが，如何でしたでしょうか。3 日間，有意義な情報交換や活発な討論が行われ，今年も充実した研究集会となりました。ご講演くださった皆様をはじめ，ご参加くださった皆様，そして関数環集会にご協力くださいました皆様に心よりお礼申し上げます。

講演者の方々に報告集原稿をご執筆頂きましたので，ここに取りまとめ報告集といたします。

世話人：三浦　毅　　　　　（新潟大学）

2016年度 関数環研究集会

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12月3日（土）

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10:00–10:45 Young Joo Lee (Chonnam National University)

演　題：**Kernels of Toeplitz operators on the Hardy space of the bidisk**

11:00–11:45 Hong Rae Cho (Pusan National University)

演　題：**The Bergman metric and related Bloch spaces on the exponentially weighted Bergman space**

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de Leeuw-Rudin's condition and products of analytic self-maps of the unit disk

茨城大学工学部　　細川　卓也 (Takuya Hosokawa)

Let $H(\mathbb{D})$ be the space of all analytic functions on the open unit disk $\mathbb{D}$ and $S(\mathbb{D})$ be the set of all analytic self-maps of $\mathbb{D}$. Let H^∞ be the set of all bounded analytic functions on $\mathbb{D}$ and U_{H^∞} be the closed unit ball of H^∞. de Leeuw and Rudin proved that an analytic self-map φ of $\mathbb{D}$ is an extreme point of U_{H^∞} if and only if

$$\int_0^{2\pi} \log \frac{1}{1-|\varphi(e^{i\theta})|} d\theta = \infty. \tag{1}$$

We will consider the following:

Question 1 *For which pair of two analytic self-maps φ, ψ does the product $\varphi \cdot \psi$ hold de Leeuw and Rudin's condition* (1)*?*

To do this, we prepare some notations.

Definition 2 *(i) Denote by $\operatorname{Ext} U_{H^\infty}$ the set of all extreme points of U_{H^∞}.*

(ii) For $\varphi \in S(\mathbb{D})$, define that $E_\varphi = \{\psi \in S(\mathbb{D}) : \varphi \cdot \psi \in \operatorname{Ext} U_{H^\infty}\}$.

We can prove the followings immediately.

Proposition 3 *Let φ be an analytic self-map of $\mathbb{D}$.*

(i) For any $\varphi \in S(\mathbb{D})$, $E_\varphi \subset \operatorname{Ext} U_{H^\infty}$.

(ii) If $\varphi \notin \operatorname{Ext} U_{H^\infty}$, then $E_\varphi = \emptyset$.

(iii) Let $\varphi = I \cdot F$ be the inner-outer decomposition of φ. Then $E_\varphi = E_F$.

(iv) If φ is an inner function, then $E_\varphi = \operatorname{Ext} U_{H^\infty}$.

Here we remark that (i) and (ii) follows from the fact that $\varphi \cdot \psi$ is not extreme if either φ or ψ is not extreme, and this fact holds more general settings.

Letting $\psi = \varphi$, we consider the relation between the extremeness of φ and the extremeness of φ^2, and more general power φ^n. From the direct calculation, we get the following.

Proposition 4 *Let φ be an analytic self-map on $\mathbb{D}$. Then the following conditions are equivalent:*

(i) $\varphi \in \operatorname{Ext} U_{H^\infty}$.

(ii) $\varphi^n \in \operatorname{Ext} U_{H^\infty}$ *for some positive integer n.*

(iii) $\varphi^n \in \operatorname{Ext} U_{H^\infty}$ *for any positive integer n.*

The proposition above gives us an idea to determine the extremeness of $\varphi \cdot \psi$, that is, if ψ is enough similar to φ, then the product $\varphi \cdot \psi$ would be similar to φ^2. Hence the extremeness of $\varphi \cdot \psi$ would be distinguished. Here we use the pseudo-hyperbolic distance $\rho(a, b)$ between a and b in $\mathbb{D}$ defined by

$$\rho(a, b) = \left| \frac{a - b}{1 - \overline{a}b} \right|.$$

Then we have the following theorem.

Theorem 5 *Let φ and ψ be in $\operatorname{Ext} U_{H^\infty}$ with $\|\varphi \cdot \psi\|_\infty = 1$. If*

$$\limsup_{n\to\infty} \rho(\varphi(z_n), \psi(z_n)) < 1$$

for any sequence $\{z_n\} \subset \mathbb{D}$ such that $|\varphi(z_n) \cdot \psi(z_n)| \to 1$, then $\varphi \cdot \psi \in \operatorname{Ext} U_{H^\infty}$.

The converse of the theorem above is not true. We give an example.

Example 6 *Let φ be an extreme point of U_{H^∞} and put $\psi(z) = -\varphi(z)$. Then*

$$\limsup_{n\to\infty} \rho(\varphi(z_n), \psi(z_n)) = 1$$

for any sequence $\{z_n\} \subset \mathbb{D}$ such that $|\varphi(z_n) \cdot \psi(z_n)| \to 1$. But $\varphi \cdot \psi = -\varphi^2$ is an extreme point of U_{H^∞}.

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Surjective isometries on $C^1([0,1])$ with respect to norms derived from plane figures

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Niigata University, Takeshi Miura

Let M and N be normed linear spaces with norms $\|\cdot\|_M$ and $\|\cdot\|_N$, respectively. A mapping $S\colon M \to N$ is an *isometry*, if and only if $\|S(f)-S(g)\|_N = \|f-g\|_M$ for all $f,g \in M$. We do not assume that isometries are linear; If S is a linear map, then S is an isometry if and only if it preserves the norm in the sense that $\|S(f)\|_N = \|f\|_M$ for all $f \in M$.

The research of isometries dates back to 1932. Let $C_{\mathbb{R}}(K)$ be the Banach space of all *real valued* continuous functions on a compact Hausdorff space K with respect to point wise operations and the supremum norm $\|f\|_\infty = \max_{k\in K}|f(k)|$ for $f \in C_{\mathbb{R}}(K)$.

Theorem 1 (Banach [1, Theorem 3 in Chapter XI]). *Let X and Y be compact metric spaces. If S is a surjective isometry, then there exist a continuous function $u\colon Y \to \{\pm 1\}$ and a homeomorphism $\phi\colon Y \to X$ such that*

$$S(f)(y) = S(0)(y) + u(y)f(\phi(y)) \qquad (\forall f \in C_{\mathbb{R}}(Y)).$$

The above statement is what Banach actually proved, and it is different from [1, Theorem 3 in Chapter XI]. It is easy to see that the opposite implication of Theorem 1 is valid.

Stone [7] obtained the same result without assuming metrizability of compact spaces.

Theorem 2 (Stone [7, Theorem 83]). *Let X and Y be compact Hausdorff spaces. If S is a surjective isometry, then there exist a continuous function $u\colon Y \to \{\pm 1\}$ and a homeomorphism $\phi\colon Y \to X$ such that*

$$S(f)(y) = S(0)(y) + u(y)f(\phi(y)) \qquad (\forall f \in C_{\mathbb{R}}(Y)).$$

The research of isometries has been extended in various directions. We will focus on $C^1([0,1])$, the complex linear space of all continuously differentiable functions on the closed unit interval $[0,1]$. There are several norms that make $C^1([0,1])$ Banach spaces. For example, the following three are typical norms on $C^1([0,1])$:

$$\|f\|_C = \sup_{t\in[0,1]}(|f(t)|+|f'(t)|), \qquad \|f\|_\Sigma = \|f\|_\infty + \|f'\|_\infty, \qquad \|f\|_\sigma = |f(0)| + \|f'\|_\infty,$$

for $f \in C^1([0,1])$. Here, $\|\cdot\|_\infty$ denotes the supremum norm on $[0,1]$, i.e. $\|g\|_\infty = \sup_{t\in[0,1]} |g(t)|$ for continuous function g on $[0,1]$. Isometries on $C^1([0,1])$ are characterized with respect to those three norms:

Theorem 3 (Cambern [2])**.** *Let S be a surjective, complex linear isometry on $C^1([0,1])$ with respect to $\|\cdot\|_C$. There exists a constant $c \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ such that*

$$S(f)(t) = cf(t) \qquad (\forall f \in C^1([0,1]),\ \forall t \in [0,1]),$$

or

$$S(f)(t) = cf(1-t) \qquad (\forall f \in C^1([0,1]),\ \forall t \in [0,1]).$$

Theorem 4 (Rao and Roy [6])**.** *Let S be a surjective, complex linear isometry on $C^1([0,1])$ with respect to $\|\cdot\|_\Sigma$. There exists a constant $c \in \mathbb{T}$ such that*

$$S(f)(t) = cf(t) \qquad (\forall f \in C^1([0,1]),\ \forall t \in [0,1]),$$

or

$$S(f)(t) = cf(1-t) \qquad (\forall f \in C^1([0,1]),\ \forall t \in [0,1]).$$

Theorem 5 (Koshimizu [4])**.** *Let S be a surjective, complex linear isometry on $C^1([0,1])$ with respect to $\|\cdot\|_\sigma$. There exist a constant $c \in \mathbb{T}$, a continuous function $u\colon [0,1] \to \mathbb{T}$ and a homeomorphism $\phi\colon [0,1] \to [0,1]$ such that*

$$S(f)(t) = cf(0) + \int_0^t u(s)f'(\phi(s))\,ds \qquad (\forall f \in C^1([0,1]),\ \forall t \in [0,1]).$$

We now recall that the Theorems 1 and 2 characterize surjective isometries without assuming their linearity. It is a natural question whether or not similar results to Theorems 3, 4 and 5 are valid for surjective, not necessarily linear, isometries. To answer this question, the Mazur-Ulam theorem plays a crucial role:

Theorem 6 (Mazur and Ulam [5])**.** *Let M and N be normed linear spaces. If $S\colon M \to N$ is a surjective isometry, then $S - S(0)$ is real linear.*

If $S\colon M \to N$ is a surjective isometry, then so is the linear map $S - S(0)$.

In Theorems 3, 4, 5, the authors characterize surjective complex linear isometries with respect to different norms. To unify these results, we introduce a norm on $C^1([0,1])$.

Definition . Let D be a compact connected subset of $[0,1] \times [0,1]$. We define

$$\|f\|_{\langle D\rangle} = \sup_{(s,t)\in D} (|f(s)| + |f'(t)|)$$

for each $f \in C^1([0,1])$.

Remark . It is easy to see that $\|\cdot\|_{\langle D\rangle}$ is a norm on $C^1([0,1])$ if and only if $p_1(D) \cup p_2(D) = [0,1]$, where p_j denotes the projection from D to the j-th coordinate in $[0,1]$.

Example . (1) If $D_1 = \{(s,t) \in [0,1] \times [0,1] : s = t\}$, then $\|f\|_{\langle D_1 \rangle} = \|f\|_C$ for all $f \in C^1([0,1])$.

(2) If $D_2 = [0,1] \times [0,1]$, then $\|f\|_{\langle D_2 \rangle} = \|f\|_\Sigma$ for all $f \in C^1([0,1])$.

(3) If $D_3 = \{(s,t) \in [0,1] \times [0,1] : s = 0\}$, then $\|f\|_{\langle D_3 \rangle} = \|f\|_\sigma$ for all $f \in C^1([0,1])$.

(4) If $D_4 = \{(s,t) \in [0,1] \times [0,1] : 0 \le s \le \frac{1}{2},\ \frac{1}{2} \le t \le 1\}$, then $\|\cdot\|_{\langle D_4 \rangle}$ is a norm on $C^1([0,1])$.

We give the characterization of surjective isometries, which need not be linear, on $C^1([0,1])$ with respect to $\|\cdot\|_{\langle D \rangle}$. This generalizes and unifies Theorems 3, 4 and 5.

Theorem 7 ([3]). *Let D be a compact connected subset of $[0,1] \times [0,1]$ such that $p_1(D) \cup p_2(D) = [0,1]$. Let $p_1(D) = [a,b]$ and $p_2(D) = [c,d]$ with $a \le b$ and $c \le d$. If $S \colon C^1([0,1]) \to C^1([0,1])$ is a surjective isometry, then there exist continuous functions $\kappa, \beta \colon [0,1] \to \mathbb{C}$, constants $\varepsilon_0, \varepsilon_1 \in \{\pm 1\}$, a C^1-diffeomorphism $\varphi \colon [a,b] \to [a,b]$ and a homeomorphism $\psi \colon [c,d] \to [c,d]$ such that $|\kappa| = 1$ on $[a,b]$, κ is constant on $[c,d]$, $|\beta| = 1$ on $[c,d]$ and*

$$S(f)(t) = S(0)(t) + \kappa(t)[f(\varphi(t))]^{\varepsilon_0} \qquad (\forall f \in C^1([0,1]), \forall t \in [a,b]),$$

$$(S(f))'(t) = (S(0))'(t) + \beta(t)[f(\psi(t))]^{\varepsilon_0 \varepsilon_1} \qquad (\forall f \in C^1([0,1]), \forall t \in [c,d]).$$

Here, we define $[f(s)]^\varepsilon$ by

$$[f(s)]^\varepsilon = \begin{cases} f(s) & \text{if } \varepsilon = 1, \\ \overline{f(s)} & \text{if } \varepsilon = -1 \end{cases}$$

for $s \in [0,1]$ and $\varepsilon \in \{\pm 1\}$.

If, in addition, $a < b$, then $\varepsilon_1 = 1$ and $\varphi = \psi$ on $[a,b] \cap [c,d]$, and there exists a constant $\gamma \in \{\pm 1\}$ such that

$$\varphi' = \gamma, \quad \beta = \kappa\gamma \quad \text{on} \quad [a,b] \cap [c,d].$$

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Weak multiplicative conditions for weighted composition operators between function algebras

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1　準備

本稿では X, Y を局所コンパクトハウスドルフ空間, $C_0(X)$ を X 上の複素数値連続関数で

$$^{\forall}\varepsilon > 0,\ ^{\exists}K \subset X\text{: コンパクト s.t. } |f(x)| < \varepsilon, ^{\forall}x \in X \setminus K$$

をみたすもの全体とする. ノルムを $\|f\|_\infty = \sup_{x \in X} |f(x)|$ で定義する.

部分集合 $A \subset C_0(X)$ が以下の条件をみたすとき, A を X 上の function algebra と呼ぶ. ：

(1) A is an algebra of $C_0(X)$ and closed under the norm $\|f\|_\infty$

(2) A separates strongly the points of X

$\overset{\text{def}}{\Longleftrightarrow}$ (i)$^{\forall}x \in X, ^{\exists}f \in A$ with $f(x) \neq 0$ and

(ii)$^{\forall}x, y \in X$ with $x \neq y$, $^{\exists}f \in A$ with $f(x) \neq f(y)$.

また, $f \in C_0(X)$ の値域の部分集合 $\sigma_\pi(f) = \{f(x) : x \in X, |f(x)| = \|f\|_\infty\}$ を f の末梢スペクトル（peripheral spectrum）という. 末梢スペクトルが 1 だけである関数を peak function と呼び, x で最大絶対値を取る peak function 全体の集合を $P(x) = \{u \in C_0(X) :\ \sigma_\pi(u) = \{1\} = \{u(x)\}\}$ と表すことにする. 更に, 部分集合 A に属する peak function の集合 $P(x) \cap A$ は $P_A(x)$ で表す. また, $x \in X$ について $|u(\xi)| < 1(^{\forall}\xi \neq x)$ となるような $u \in P_A(x)$ が存在するとき, x は A の peak point と呼び, ある $P_A(x)$ の関数族 $\{u_\alpha\}$ で $\{x\} = \cap_{u \in \{u_\alpha\}} u^{-1}(\{1\})$ と表せるとき, x は A の weak peak point と呼ぶ.

以下, A, B を X, Y 上の function algebra で X は A の weak peak point 全体の集合, Y は B の weak peak point 全体の集合と一致することとする. また, T を A から B への全射とする.

2　積のスペクトルを保存する写像とその一般化

2001 年に Molnár は以下の結果を示した.

Theorem (Molnár, 2001 [6]). X が第一可算公理をみたし更にコンパクトとする. X 上の複素数値連続関数全体からなるバナッハ環を $C(X)$, $\sigma(\cdot)$ をスペクトルとする. $C(X)$ からそれ自身への全射 S が $\sigma(S(f)S(g)) = \sigma(fg)$ ($^{\forall} f, g \in C(X)$) をみたすとき, 同相写像 $\phi : X \to X$ と連続関数 $^{\exists}\, \alpha : X \to \{1, -1\}$ が存在して $S(f) = \alpha \cdot (f \circ \phi)$($^{\forall} f \in C(X)$) と表せる. つまり S は荷重合成作用素である.

この結果では, 乗法性や線形性を仮定していない. 積の値域が変わらない, という仮定のみからその写像の構造を完全に決定しており, 一つの荷重合成作用素の特徴づけを与えているとも言える. これは大変興味深いものであり, 多くの数学者によって更なる研究が行われている. 特に, function algebra に関する結果を以下にいくつか紹介する. なお, 実際の論文のほとんどではここで紹介するものより一般的な結果が示されている. しかし今回は先に仮定した前提の場合に限定した結果として紹介することとする.

Theorem (Rao-Roy, 2005 [7]). $A = B$, $\sigma(T(f)T(g)) = \sigma(fg)$ ($^{\forall} f, g \in A$) をみたすとき, 同相写像 $\phi : X \to X$ と連続関数 $^{\exists}\, \alpha : X \to \{1, -1\}$ が存在して $T(f) = \alpha \cdot (f \circ \phi)$($^{\forall} f \in A$) と表せる.

Theorem (Honma, 2006 [3]). $A = C_0(X), B = C_0(Y), \sigma(T(f)T(g)) = \sigma(fg)$ ($^{\forall} f, g \in A$) をみたすとき, 同相写像 $\phi : Y \to X$ と連続関数 $^{\exists}\, \alpha : Y \to \{1, -1\}$ が存在して $T(f) = \alpha \cdot (f \circ \phi)$($^{\forall} f \in A$) と表せる.

Theorem (Hatori-Miura-Oka-Takagi, 2009 [2]). $\sigma_\pi(T(f)T(g)) = \sigma_\pi(fg)$ ($^{\forall} f, g \in A$) をみたすとき, 同相写像 $\phi : Y \to X$ と連続関数 $^{\exists}\, \alpha : Y \to \{1, -1\}$ が存在して $T(f) = \alpha \cdot (f \circ \phi)$($^{\forall} f \in A$) と表せる.

Theorem (Tonev, 2010 [8]). $\sigma_\pi(T(f)) = \sigma_\pi(f), \sigma_\pi(T(f)T(g)) \cap \sigma_\pi(fg) \neq \emptyset$ ($^{\forall} f, g \in A$) をみたすとき, 同相写像 $\phi : Y \to X$ と連続関数 $^{\exists}\, \alpha : Y \to \{1, -1\}$ が存在して $T(f) = \alpha \cdot (f \circ \phi)$($^{\forall} f \in A$) と表せる.

上記で紹介した結果は, Molnár による結果で用いられている値域の集合をより小さい部分集合に制限した上でも同様な結論が導かれることを示している. この流れを受けて, 荷重合成作用素を特徴付ける集合はどこまで小さくできるのか？という疑問が浮かぶ. 2010 年の Tonev の結果を見ると, かなり一般的な条件にまで達していると思われるが, それでも付加条件があり, まだ改良できる余地があると考えられる. この点について未だ完全な解答は得られていない. しかし, 別の付加条件を与えることで一つの条件式のみから同様な結果を得ることが出来た. 本稿ではその得られた結果を証明の概略とあわせて紹介する.

3 主結果と証明の概略

今回得られた結果は以下である.

Theorem 1 (T.). T は $\sigma_\pi(T(f)T(g)) \cap \sigma_\pi(fg) \neq \emptyset$ $({}^\forall f, g \in A)$ をみたすものとする. X が第一可算公理をみたすとき, 同相写像 $\phi : Y \to X$ と連続関数 ${}^\exists\ \alpha : Y \to \{1, -1\}$ が存在して $T(f) = \alpha \cdot (f \circ \phi)({}^\forall f \in A)$ と表せる.

なお, 単位元を持つ uniform algebra に関する同様な結果は [4] でも示されている.

主結果の証明には, この分野全体で随所に活用される peak function の性質に関する結果と積のノルムを保存する写像に関する結果を用いる.

Lemma 2. $x \in X, f \in A$ に対して $f(x) \neq 0$ とする. x の任意の近傍 U_x について

$$\sigma_\pi(fu) = \{f(x)\},\ |fu(\xi)| < |f(x)| \text{ on } X \setminus U_x$$

となるような peak function $u \in P(x)$ が存在する.

特に, 以下が成り立つ.

Lemma 3. $x \in X, f \in A$ に対して $f(x) \neq 0$ とする. X が第一可算公理をみたすとき,

$$\sigma_\pi(fu) = \{f(x)\},\ |fu(\xi)| < |f(x)| \text{ on } X \setminus \{x\}$$

となるような peak function $u \in P(x)$ が存在する.

Theorem 4 [1, 5, 8]. $\|T(f)T(g)\|_\infty = \|fg\|_\infty$ $({}^\forall f, g \in A)$ をみたすとき, 同相写像 $\phi : Y \to X$ が存在して $|T(f)| = |f \circ \phi|({}^\forall f \in A)$ と表せる.

(同相写像 ϕ の構成) ${}^\forall y \in Y$ を固定する. ${}^\forall u \in T^{-1}(P_B(y))$ に対して $|u|^{-1}(1) \overset{\text{def}}{=} u^{-1}(\sigma_\pi(u))$ とする. この $|u|^{-1}(1)$ の共通部分 $\cap_{u \in T^{-1}(P_B(y))} |u|^{-1}(1)$ は 1 点のみからなる集合 $\{x_y\}$ である. $\phi(y) \overset{\text{def}}{=} x_y$ と定義する. すると, この ϕ は Y から X への同相写像で $|T(f)| = |f \circ \phi|({}^\forall f \in A)$ をみたす.

(Theorem 1 の証明) 条件と Theorem 4 より同相写像 $\phi : Y \to X$ が存在して

$$|T(f)(y)| = |f(\phi(y))|({}^\forall f \in A, {}^\forall y \in Y)$$

をみたす. したがって

$$T(f)(y) = 0 \Leftrightarrow f(\phi(y)) = 0$$

である.

以下, $T(f)(y)f(\phi(y)) \neq 0$ と仮定する. X は第一可算公理をみたすから, Lemma 3 より

$$\sigma_\pi(fu) = \{f(\phi(y))\},\ |fu(x)| < |f(\phi(y))| \text{ on } X \setminus \{\phi(y)\}$$

となる $u \in P_A(\phi(y))$ が存在する.

$$\sigma_\pi(T(f)T(u)) \cap \sigma_\pi(fu) = \sigma_\pi(T(f)T(u)) \cap \{f(\phi(y))\} \neq \emptyset$$

であるから, $T(f)(y')T(u)(y') = f(\phi(y))$ となる $y' \in Y$ が存在する.

$$|f(\phi(y))| = |T(f)(y')T(u)(y')| = |f(\phi(y'))u(\phi(y'))| = |(fu)(\phi(y'))|$$

より $y' = y$ で更に $T(f)(y)T(u)(y) = f(\phi(y))$ となる. 特に $^\forall v \in P_A(\phi(y))$ について $f = v, u$ とすると

$$T(v)(y)T(u)(y) = 1, T(u)(y)^2 = 1$$

であるから

$$T(v)(y) = T(u)(y)$$

となり, $T(v)(y)$ は各 y に関して一意に定まる.

$$\alpha(y) \stackrel{\text{def}}{=} T(v)(y)$$

と定義すると $\alpha(y)^2 = 1$ である. 以上より,

$$T(f)(y) = T(f)(y)T(u)(y)^2 = T(u)(y)f(\phi(y)) = \alpha(y)f(\phi(y))(^\forall f \in A, ^\forall y \in Y)$$

がわかる. $T(f), f \circ \phi$ は連続より α は連続である.

□

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Research on Fejér-Riesz type inequalities for Bergman spaces

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First, we set several notations.

$$D := \{z \in \mathbb{C} : |z| < 1\}.$$

$$\partial D := \{z \in \mathbb{C} : |z| = 1\}.$$

$$H(D) := \{f : f \text{ is analytic in } D\}.$$

Let $0 < p < \infty$.

$$H^p := \left\{ f \in H(D) : \|f\|_{H^p} := \sup_{r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{\frac{1}{p}} < \infty \right\}.$$

If $p = \infty$,

$$H^\infty := \left\{ f \in H(D) : \|f\|_{H^\infty} := \sup_{|z|<1} |f(z)| < \infty \right\}.$$

H^p are called Hardy spaces. If $0 < p < 1$, then H^p is a complete metric space under the metric $d(f,\, g) := \|f - g\|_{H^p}^p$. If $1 \le p \le \infty$, then H^p is a Banach space under the norm $\|f\|_{H^p}$. Here, we would like to introduce original Fejer-Riesz inequality for the Hardy spaces H^p.

Theorem 1 (Fejér-Riesz inequality for H^p, [2]) *Let $0 < p < \infty$. If $f \in H^p$, then*

$$\int_{-1}^{1} |f(x)|^p dx \le \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta \quad \left(= \pi \int_0^{2\pi} |f(e^{i\theta})|^p \frac{d\theta}{2\pi} = \pi \|f\|_{H^p}^p \right).$$

The constant $\frac{1}{2}$ is best possible.

Let $0 < p < \infty$ and $-1 < \alpha < \infty$.

$$A^p_\alpha := \left\{ f \in H(D) : \|f\|_{A^p_\alpha} := \left(\int_D |f(z)|^p (1+\alpha)(1-|z|^2)^\alpha dA(z) \right)^{\frac{1}{p}} < \infty \right\}.$$

where $dA(z) = \dfrac{1}{\pi} dxdy = \dfrac{1}{\pi} r dr d\theta \quad (z = x + iy = re^{i\theta})$.

A^p_α are called weighted Bergman spaces. If $0 < p < 1$, then A^p_α is a complete metric space under the metric $d(f,\, g) := \|f - g\|_{A^p_\alpha}^p$. If $1 \le p < \infty$, then A^p_α is a Banach space under the norm $\|f\|_{A^p_\alpha}$.

In 2012, Andreev proved that the Fejér-Riesz type inequalities hold for the weighted Bergman spaces A^2_α.

Theorem 2 (Fejér-Riesz type inequality for A^2_α, [1]) *Let $-1 < \alpha < \infty$. If $f \in A^2_\alpha$, then for any $\zeta \in \partial D$,*

$$\int_0^1 |f(\zeta x)|^2 (1-x)^{1+\alpha} dx \le \lambda_\alpha \int_D |f(z)|^2 (1+\alpha)(1-|z|^2)^\alpha dA(z) \ \left(= \lambda_\alpha \|f\|^2_{A^2_\alpha}\right).$$

Here $\lambda_\alpha \le \dfrac{1}{\pi^\alpha}$ (if $-1 < \alpha < 0$), $\lambda_\alpha \le \dfrac{1}{1+\alpha}$ (if $0 \le \alpha$).

Yonezawa Mathematics Seminar in 2014, Kazuhiro Kasuga gave a lecture about Andreev's results above, and he conjectured that Fejer-Riesz type inequalities for the weighted Bergman spaces A^p_α $(0 < p < \infty)$ would hold:

Conjecture Let $0 < p < \infty$ and $-1 < \alpha < \infty$. If $f \in A^p_\alpha$, then for any $\zeta \in \partial D$, does

$$\int_0^1 |f(\zeta x)|^p (1-x)^{1+\alpha} dx \le \lambda_\alpha \int_D |f(z)|^p (1+\alpha)(1-|z|^2)^\alpha dA(z) \ \left(= \lambda_\alpha \|f\|^p_{A^p_\alpha}\right)$$

hold?

We can get a trivial partial ansewer. Let $0 < p < \infty$. Suppose that $f \in A^p_\alpha$ has no zeros in D. Then we can define $\{f(z)\}^{\frac{p}{2}}$, and $\{f(z)\}^{\frac{p}{2}} \in A^2_\alpha$. Applying Andreev's result to $\{f(z)\}^{\frac{p}{2}}$, we have

$$\int_0^1 |f(\zeta x)|^p (1-x)^{1+\alpha} dx \le \lambda_\alpha \int_D |f(z)|^p (1+\alpha)(1-|z|^2)^\alpha dA(z). \tag{1}$$

If $f \in A^p_\alpha$ has zeros in D, I do not know whether inequalities (1) hold.

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Another approach to a self-adjointness of operators with a Kato-Rellich potential

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1 Introduction

We have been studied (unbounded and linear) semiclosed operators in a Hilbert space H. Motivations for studying such operators are the following.

· The set $\mathcal{S}(H)$ of semiclosed operators in H is well behavior class in a sense. That is, $\mathcal{S}(H)$ is closed under sums, products, (weak) adjoints and closures if they exist.
· The set $\mathcal{S}(H)$ contains the set $\mathcal{B}(H)$ of bounded operators and the set $\mathcal{C}(H)$ of closed operators. Hence, $s + t \in \mathcal{S}(H)$ if $s, t \in \mathcal{C}(H)$.
· A semiclosed operator is equivalent to a quotient of bounded operators. Hence, we can use some technique of bounded operators.
· The set $\mathcal{S}(H)$ is metrizable.

It is known that the set $\mathcal{S}(H)$ is metrizable by the q-metric which is introduced in [1]. Let $\mathcal{S}_{sym}(H)$ and $\mathcal{S}_{sa}(H)$ be the set of semiclosed symmetric operators and selfadjoint operators, respectively. Then we have a result ([2]) that $\mathcal{S}_{sa}(H)$ is relatively open in $\mathcal{S}_{sym}(H)$:

$$(\mathcal{S}(H), q) \supset \mathcal{S}_{sym}(H) \underbrace{\supset \mathcal{S}_{sa}(H)}_{\text{relatively open}}$$

Using the above result, we deduce a self-adjointness of Schrödinger operators with a Kato-Rellich potential, which is well known as Kato's Theorem. To give another proof for such the theorem is main purpose in this note. These are also stated in [2].

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2 Kato's Theorem

A real valued function $V(x)$ on $\mathbb{R}^N$ is said to be a Kato-Rellich potential if it is decomposed by

$$V = V_1 + V_2 \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N).$$

Here, $p = 2$ if $N = 1, 2, 3$ and some p $(> \frac{N}{2})$ if $N \geq 4$.

The following function is an example of Kato-Rellich potentials. $V(x) = \frac{c}{|x|^k}$ $(x \in \mathbb{R}^N)$, where c is a constant, $0 < k < \frac{N}{2}$ if $N = 1, 2, 3$ and $0 < k < 2$ if $N \geq 4$.

The following is Kato's theorem.

Theorem 2.1 *Let V be a Kato-Rellich potential on $\mathbb{R}^N$ $(N \geq 1)$. Then, $-\Delta + V$ is a self-adjoint operator with a maximal domain* $\mathrm{dom}(-\Delta)$ *in* $L^2(\mathbb{R}^N)$.

3 Preliminaries

Let $(H, (\ ,\ ))$ be an infinite dimensional complex Hilbert space. A subspace M in H is said to be semiclosed if there exists an inner product $(\cdot, \cdot)_M$ on M such that $(M, \|\cdot\|_M)$ is a Hilbert space and the inclusion mapping $(M, \|\cdot\|_M) \hookrightarrow H$ is continuous. Clearly, a closed subspace is a semiclosed subspace. As another typical example, Sobolev spaces in $L^2(\mathbb{R}^N)$ are semiclosed subspaces in $L^2(\mathbb{R}^N)$. A subspace M in H is semiclosed if and only if M is a bounded operator range in H, that is, $M = XH$ for some $X \in \mathcal{B}(H)$. An operator $s : \mathrm{dom}(s) \to H$ is said to be semiclosed (resp. closed), if the graph $\{(u, su) \in H \times H : u \in \mathrm{dom}(s)\}$ of s is semiclosed (resp. closed) in the product Hilbert space $H \times H$. Clearly, a closed operator is a semiclosed operator. A semiclosed operator is equivalent to a quotient of bounded operators. That is, an operator s belongs to $\mathcal{S}(H)$ if and only if

$$s = Y/X : Xu \to Yu,\ (u \in H) \text{ for some } X, Y \in \mathcal{B}(H)$$

with $\ker X \subseteq \ker Y$. A densely defined operator $s : \mathrm{dom}(s) \to H$ is said to be symmetric, if $(su, v) = (u, sv)$ $u, v \in \mathrm{dom(s)}$. Namely,

$$su = s^*u, \quad u \in \mathrm{dom(s)} \subseteq \mathrm{dom(s^*)}.$$

Simply, symmetric operator s is denoted by $s \subseteq s^*$. When the equation $s = s^*$ holds, s is said to be selfadjoint.

4 A choice function α and the q-metric for $\mathcal{S}(H)$

We choose a Hilbert norm $\|\cdot\|_M$ from each semiclosed subspace M. Denote such a choice function by α.

Now, let α be a choice function as above. For a given semiclosed subspace M, there exists a Hilbert norm $\|\cdot\|_M$ by α such that $(M, \|\cdot\|_M) \hookrightarrow H$. It is known that a Hilbert space $(M, \|\cdot\|_M)$

is isometrical isomorphic to de Branges space $\mathcal{M}(A)$ for some $A \geq 0$ in $\mathcal{B}(H)$. To choose a Hilbert norm is equivalent to choose a positive operator $A \geq 0$ satisfying $M = AH$ and $\|\cdot\|_M = \|\cdot\|_A$. Here de Branges norm $\|\cdot\|_A$ is defined by $(Ax, Ay)_A := (Px, Py)$, $(x, y \in H)$, P is the orthogonal projection onto $(\ker A)^\perp$. Therefore the notation α has two meanings such as

$$\begin{cases} \cdot \text{ a choice of Hilbert norms} \\ \cdot \text{ a choice of positive operators.} \end{cases}$$

We introduce a metric in the set $\mathcal{S}(H)$ of semiclosed operators. For semiclosed operators $s, t \in \mathcal{S}(H)$, since domains $\mathrm{dom}(s)$ and $\mathrm{dom}(t)$ are semiclosed, there exist positive bounded operators A and C such that $\mathrm{dom}(s) \overset{\alpha}{=} AH$ and $\mathrm{dom}(t) \overset{\alpha}{=} CH$. Hence they are uniquely represented by quotients of bounded operators :

$$s \overset{\alpha}{=} B/A, \quad t \overset{\alpha}{=} D/C,$$

where $A, C \in \mathcal{B}^+(H)$, $B, D \in \mathcal{B}(H)$ with $\ker A \subseteq \ker B$ and $\ker C \subseteq \ker D$. Then we define the metric between s and t by

Definition 4.1 $\quad q(s,t)(= q_\alpha(s,t)) := \max\{\|A - C\|, \|B - D\|\}.$

5 A radius of Laplacian $k\Delta$

Let $\mathcal{S}_{sym}(H)$ be the set of (densely defined) semiclosed symmetric operators, and let $\mathcal{S}_{sa}(H)$ be the set of selfadjoint operators.

$$(\mathcal{S}(H), q) \supset \mathcal{S}_{sym}(H) \supset \mathcal{S}_{sa}(H).$$

Theorem 5.1 ([2]) *The set $\mathcal{S}_{\mathrm{sa}}(H)$ is relatively open in $\mathcal{S}_{\mathrm{sym}}(H)$. That is, $\forall s \in \mathcal{S}_{sa}(H)$, $\exists \delta > 0$ such that*

$$q(s,t) < \delta \text{ and } t \in \mathcal{S}_{sym}(H) \text{ imply } t \in \mathcal{S}_{sa}(H).$$

We do not give a proof of Theorem 5.1 here. But we clarify a radius δ (we call δ in Theorem 5.1 a radius of s) as the following. Let $s \in \mathcal{S}_{sa}(H)$, $s \overset{\alpha}{=} B/A$ and $R := (A^2 + B^*B)^{\frac{1}{2}}$. Then, a radius of s is given by

$$\delta = \frac{1}{2}\|R^{-1}\|^{-1} = \frac{1}{2}\gamma(R), \tag{1}$$

where $\gamma(R) = \inf\{\|Rf\| : f \in (\ker R)^\perp, \|f\| = 1\}$. Then we have a question.

What is a radius δ of Laplacian $k\Delta$ in $L^2(\mathbb{R}^N)$? $\quad (k \in \mathbb{R} \setminus \{0\})$

Now, we will reply this question. Let $k\Delta \overset{\alpha}{=} kB/A$ be a quotient. The domain of $k\Delta$ is the Sobolev space $H^2(\mathbb{R}^N)$ with the order 2 which is isometrical isomorphic to de Branges space $\mathcal{M}(A)$, $(A = (I - \Delta)^{-1})$ (cf. [1]). Hence we see that

$$R = (A^2 + (kB)^*(kB))^{\frac{1}{2}} = (A^2 + k^2B^*B)^{\frac{1}{2}}.$$

Then,

$$\begin{aligned}
\gamma(R)^2 &= \inf_{\|f\|=1} \|Rf\|^2 = \inf_{\|f\|=1} \|(A^2 + k^2 B^* B)^{\frac{1}{2}} f\|^2 = \inf_{\|f\|=1} \{\|Af\|^2 + \|kBf\|^2\} \\
&= \inf_{\|f\|=1} \{\|(I - \Delta)^{-1} f\|^2 + \|k\Delta(I - \Delta)^{-1} f\|^2\} \\
&= \inf_{\|\widehat{f}\|=1} \{\|(1 + |\xi|^2)^{-1} \widehat{f}\|^2 + \|k|\xi|^2 (1 + |\xi|^2)^{-1} \widehat{f}\|^2\} \\
&= \inf_{\|\widehat{f}\|=1} \Big\| \frac{(1 + k^2|\xi|^4)^{\frac{1}{2}}}{1 + |\xi|^2} \widehat{f} \Big\|^2 = \inf_{\|\widehat{f}\|=1} \|M\widehat{f}\|^2 = \gamma(M)^2, \quad ((\ker M)^\perp = L^2_\xi(\mathbb{R}^N)),
\end{aligned}$$

where $M := \dfrac{(1 + k^2|\xi|^4)^{\frac{1}{2}}}{1 + |\xi|^2}$ is the multiplication operator in $L^2_\xi(\mathbb{R}^N)$. Hence

$$\begin{aligned}
\delta &= \frac{1}{2}\gamma(R) = \frac{1}{2}\gamma(M) = \frac{1}{2}\|M^{-1}\|^{-1} = \frac{1}{2}\Big(\Big\| \frac{1 + |\xi|^2}{(1 + k^2|\xi|^4)^{\frac{1}{2}}} \Big\|_\infty\Big)^{-1} \\
&= \frac{1}{2}\Big(\Big\| \frac{(1 + |\xi|^2)^2}{1 + k^2|\xi|^4} \Big\|_\infty\Big)^{-\frac{1}{2}} = \frac{1}{2}\Big(\frac{1 + k^2}{k^2}\Big)^{-\frac{1}{2}}.
\end{aligned}$$

Proposition 5.2 *A radius δ of $k\Delta$ is $\delta = \dfrac{|k|}{2\sqrt{1 + k^2}}$. In paticular, a radius of $-\Delta$ is $\dfrac{\sqrt{2}}{4}$.*

Remark 5.1 *From a view point of quantum mechanics, Shrödinger operator is given by a form of $-\frac{\hbar^2}{2m}\Delta + V$, where m is mass of a particle and $\hbar$ is $h/2\pi$ for the planck constant h. This is a case of $k = -\frac{\hbar^2}{2m}$ in a radius formula, so that δ of $-\frac{\hbar^2}{2m}\Delta$ is*

$$\delta = \frac{|k|}{2\sqrt{1 + k^2}} = \frac{\hbar^2}{2\sqrt{4m^2 + \hbar^4}}.$$

A value of this expression seems to be dependent on the choice of physical units. If so, it is slightly mysterious.

6 Another approach to a self-adjointness of operators with a Kato-Rellich potential

Based on Theorem 5.1 and Proposition 5.2 , we will show a self-adjointness of the Schrödinger operator $-\Delta + V$ with a Kato-Rellich potential V, $V = V_1 + V_2 \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$. Here, $p = 2$ if $N = 1, 2, 3$ and some $p > \frac{N}{2}$ if $N \geq 4$. Clearly $\mathrm{dom}(V) = \mathrm{dom}(V_1) := \{f \in L^2(\mathbb{R}^N) : V_1 f \in L^2(\mathbb{R}^N)\}$. By Sobolev embedding theorem as below, we see that $\mathrm{dom}(-\Delta) \subseteq \mathrm{dom}(V_1)$:

$$V_1 f \in L^2(\mathbb{R}^N) \quad \text{if} \quad \forall f \in H^2(\mathbb{R}^N).$$

Theorem 6.1 (Sobolev embedding theorem) *The following assertions hold.*

(i) $H^2(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ *if* $N = 1, 2, 3$.

(ii) $H^2(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ $0 < \forall q < (\frac{1}{2} - \frac{2}{N})^{-1}$ *if* $N \geq 4$.

Hence we have

$$\mathrm{dom}(-\Delta + V) = \mathrm{dom}(-\Delta + V_1 + V_2) = \mathrm{dom}(-\Delta + V_1) = \mathrm{dom}(-\Delta).$$

First, we explain an outline of approach for selfadjointness. For $V = V_1 + V_2 \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, we can find sequences

$$\{V_{i,n}\}_{n=1}^{\infty} \text{ for } i = 1, 2 \text{ such that } V = V_{1,n} + V_{2,n} \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$$
$$\text{with } \|V_{1,n}\|_{L^p} \to 0 \text{ as } n \to \infty.$$

Then,

· We easily see $-\Delta + V \in \mathcal{S}_{sym}(H)$.

· We see that $-\Delta + V_{2,n}$ is selfadjoint on $\mathrm{dom}(-\Delta)$.

· We see that, by lemma 6.3, a radius of $-\Delta + V_{2,n}$ is taken as the same of a radius of $-\Delta$.

· $q(-\Delta + V, -\Delta + V_{2,n}) \to 0$ as $n \to \infty$.

Hence, by Theorem 5.1, we conclude that $-\Delta + V$ is selfadjoint with $\mathrm{dom}(-\Delta)$.

Lemma 6.2 *For* $s, t \in \mathcal{S}(H)$ *with* $\mathrm{dom}(s) = \mathrm{dom}(t)$, *let* $s \overset{\alpha}{=} B/A$ *and* $t \overset{\alpha}{=} D/A$. *Then,*

$$q(s, t) = q(s + X, t + X),$$

for any $X \in \mathcal{B}(H)$.

Proof.

$$\begin{aligned} q(s+X, t+X) &= q\Big(B/A + X/I,\ D/A + X/I\Big) = q\Big(B/A + XA/A,\ D/A + XA/A\Big) \\ &= q\Big((B + XA)/A,\ (D + XA)/A\Big) = \|(B + XA) - (D + XA)\| \\ &= \|B - D\| = q(s, t). \quad \square \end{aligned}$$

Lemma 6.3 *Let* $s \in \mathcal{S}_{sa}(H)$ *and* $S \in \mathcal{B}_{sa}(H)$. *For a radius* δ *of* s, δ *is also a radius of selfadjoint operator* $s + S$. *That is,* $q(s + S, t) < \delta$ *and* $t \in \mathcal{S}_{sym}(H)$ *imply* $t \in \mathcal{S}_{sa}(H)$.

Proof. Suppose that $q(s + S, t) < \delta$. By Lemma 6.2, we see that

$$q(s + S, t) = q\Big(s + S + (-S), t + (-S)\Big) = q\Big(s, t + (-S)\Big) < \delta.$$

Since $t+(-S)\in\mathcal{S}_{sym}(H)$, we see that $t+(-S)\in\mathcal{S}_{sa}(H)$. Therefore, $t\ (=t+(-S)+S)$ is selfadjoint. $\square$

Let V be a Kato-Rellich potential such that

$$V=V_1+V_2\in L^p(\mathbb{R}^N)+L^\infty(\mathbb{R}^N).$$

If $V\in L^\infty$, then $-\Delta+V$ is selfadjoint. Hence, we may assumed that V is unbounded. For sufficiently large $n\in\mathbb{N}$ such that $\|V_2\|_\infty<n$, we define

$$Z_n:=\{x\in\mathbb{R}^N:|V(x)|>n\}.$$

Then we can see the following assertions.

- $V_{1,n}(x):=V_1(x)\chi_{Z_n}(x)\ \Big(=V(x)\chi_{Z_n}(x)\Big)$
- $V_{2,n}(x):=V(x)-V_{1,n}(x)\ \Big(=(V_1-V_{1,n})(x)+V_2(x)\Big).$
- $V=V_{1,n}+V_{2,n}\in L^p(\mathbb{R}^N)+L^\infty(\mathbb{R}^N)$
- $\|V_{1,n}\|_{L^p}\to 0$ as $n\to\infty$ by Lebesgue convergence theorem.

Now, let $-\Delta\overset{\alpha}{=}B/A$, where $H^2(\mathbb{R}^N)\cong\mathcal{M}(A)$. And let $V\overset{\alpha}{=}D/C$. Since $\mathrm{dom}(-\Delta)(=AH)\subseteq\mathrm{dom}(V)(=CH)$, $\exists X\in\mathcal{B}(H)$ such that $A=CX$ by Douglas's majorization theorem. Then,

$$\begin{aligned}
q(-\Delta+V,-\Delta+V_{2,n})&=q\Big(B/A+D/C,B/A+V_{2,n}/I\Big)=q\Big(B/A+DX/CX,B/A+V_{2,n}A/A\Big)\\
&=q\Big((B+DX)/A,(B+V_{2,n}A)/A\Big)=\|DX-V_{2,n}A\|\\
&=\|VCX-V_{2,n}A\|\quad(D=VC)\\
&=\|VA-V_{2,n}A\|=\|(V-V_{2,n})A\|=\|V_{1,n}A\|.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|V_{1,n}A\|&=\sup_{\|g\|=1,g\in L^2}\|V_{1,n}Ag\|=\sup_{\|Ag\|_A=1}\|V_{1,n}Ag\|=\sup_{\|f\|_{H^2}=1}\|V_{1,n}f\|\quad(f:=Ag)\\
&\le\sup_{\|f\|_{H^2}=1}\|V_{1,n}\|_{L^p}\|f\|_{L^q}\le\sup_{\|f\|_{H^2}=1}\|V_{1,n}\|_{L^p}\cdot C\|f\|_{H^2}\quad(\exists C>0)\\
&=C\|V_{1,n}\|_{L^p}\to 0\quad(n\to\infty).
\end{aligned}$$

Remark 6.1 *We apply the following relations as above.* $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$

$$H^2(\mathbb{R}^N)\hookrightarrow L^q(\mathbb{R}^N),\ \textit{where}\ \begin{cases}q=\infty & (N=1,2,3)\\ 0<q<(\frac{1}{2}-\frac{2}{N})^{-1} & (N\ge 4).\end{cases}$$

In summary, this means that $q(-\Delta+V,-\Delta+V_{2,n})\to 0\ (n\to\infty)$. On the other hand, since $V_{2,n}$ is a bounded selfadjoint operator, it follows from Lemma 6.3 that a radius of $-\Delta+V_{2,n}$ can be taken by a radius $\delta=\frac{\sqrt{2}}{4}$ of $-\Delta$. Therefore, the selfadjoint operator $-\Delta+V_{2,n}$ is sufficiently near to the semiclosed symmetric operator $-\Delta+V$ so that their distance is within $\frac{\sqrt{2}}{4}$ for large n. Hence we conclude that $-\Delta+V$ is selfadjoint.

On reducing subspaces for a class of Toeplitz operators on weighted Hardy spaces over bidisk

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1 Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ a set of bounded operators on $\mathcal{H}$. A subspace X in $\mathcal{H}$ is an invariant subspace for $A \in \mathcal{H}$ if $AX \subset X$. Moreover X is a reducing subspace if X is an invariant subspace for both A and its adjoint A^*. The reducing subspace X is called minimal if the reducing subspaces contained in X are trivial.

Stessin and Zhu [6] characterized the minimal reducing spaces for shift operator with finite multiplicity on the weighted Hardy spaces over the unit disk. Let $H^2_\omega(\mathbb{D})$ be the weighted Hardy space over the unit disk which consists of analytic functions with finite norm determined by the weight ω. Put S the shift operator on $H^2_\omega(\mathbb{D})$.The statement is as follows;

Theorem 1.1 *Let N be a natural number. Then there are N minimal reducing subspaces for S^N and therefore 2^N reducing subspaces under some appropriate condition with the weight; otherwise there are infinitely many minimal reducing subspaces for S^N.*

For example, if $H^2_\omega(\mathbb{D})$ is the Bergman space, then there are N minimal reducing subspaces for S^N which are in the form of

$$X_n = \text{Span}\ \{z^{n+kN}; k = 0, 1, 2, \ldots\} \text{ for} n = 0, 1, 2, \ldots, N-1,$$

where Span denotes the closed linear span of a subspace. Moreover there are 2^N reducing subspaces for S^N consisting of X_n's.

There are several approaches to generalizing the results in [6]. In this presentation, we consider a weighted Hardy space over bidisk. The definition are as follows; let $\omega = \{(\omega_1, \omega_2)\}$ be a set of positive numbers with

$$\sup \frac{\omega(n_1+1, n_2)}{\omega(n_1, n_2)} < \infty \text{ and } \sup \frac{\omega(n_1, n_2+1)}{\omega(n_1, n_2)} < \infty. \tag{1}$$

The weighted Hardy space $H^2_\omega(\mathbb{D}^2)$ is a Hilbert space of analytic functions over the bidisk which satisfy

$$\|f\|^2 = \sum |a(n_1, n_2)|^2 \omega(n_1, n_2) < \infty,$$

where $f(z,w) = \sum a(n_1,n_2)z^{n_1}w^{n_2}$. From the condition (1), we see that multiplication operators defined by coordinate function are bounded.

Fix natural numbers N_1 and N_2. As one of the extention of [6], we determined the reducing subspaces for multiplication operators defined by z^{N_1} and w^{N_2} in [2]. In the next section, we will see other generalizations of the results in [6].

2 Main results

Assume $H^2_\omega(\mathbb{D}^2)$ is the Bergman space over the bidisk. Lu and Zhou [5] determined the reducing subspaces for the multiplication operator defined by $z^{N_1}w^{N_1}$. Albaseer, Lu and Shi [1] determine the reducing subspaces for the Toeplitz operator defined by $z^{N_1}\overline{w}^{N_2}$.

We note that we obtain the results on general $H^2_\omega(\mathbb{D}^2)$. In this conference, Theorem 2.1 was presented; for definition of transparent polynomial of function, see the end of section 2.

Theorem 2.1 ([3]) *Let N be a natural number and $M_{z^N w^N}$ the multiplication operator defined by $z^N w^N$. For a reducing subspace X for $M_{z^N w^N}$ there is a transparent function such that the reducing subspace generated by the transparent function is contained in X. Moreover if X is minimal, then X is generated by the transparent function.*

After the conference, we obtain Theorem 2.2 which will be appeared in [4];

Theorem 2.2 ([4]) *Let N be a natural number and $T_{\overline{z}^N w}$ the Toeplitz operator defined by $\overline{z}^N w$. For a reducing subspace X for $T_{\overline{z}^N w}$ there is a transparent polynomial such that the reducing subspace generated by the transparent function is contained in X. Moreover if X is minimal, then X is generated by the transparent function.*

Roughly speaking, a function is called transparent if a part of terms in the function cannot be removed under shift operation, the adjoint of shift operation and linear operations. The transparent function or polynomial will be the generator of the minimal reducing subspaces.

3 Proof of Main results

The main idea of not only the statement but also the proof is in [6]. In this report, we present the summery of the proof. The proof consists of four steps;

(1)If X is a reducing subspace and (n_1,n_2) is the minimal multi-index of functions in X, extremal problem

$$\sup\{\mathrm{Re}\frac{\partial^{n_1+n_2}}{\partial^{n_1}z\partial^{n_2}w}f(0,0); f\in X\}$$

has a unique solution G.

(2)Each terms in the function G cannot be removed under shift operation, the adjoint of shift operation and linear operations. Therefore G is transparent.

(3)The reducing subspace generated by G is the smallest reducing subspace contained in X.

(4)Moreover if X is minimal, then X is equal to the reducing subspace generated by G.

We have some possibility of connecting the study of reducing subspaces with that of the commutants. Using the results in this report, the speaker would like to study the commutants of some operators.

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Commutativity of self-adjoint elements

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1 Introduction

Let A be a unital C^*-algebra. An element $a \in A$ is called self-adjoint if $a = a^*$. The set of all self-adjoint elements in A is denoted byA_{sa}. It is a real-linear subspace of A. A matrix in $M_n(\mathbb{C})$, the C^*-algebra of all complex $n \times n$ matrices, is self-adjoint iff it is a Hermitian matrix. If A is commutative, then the theorem of Gelfand-Naimark asserts that A is isometrically isomorphic to $C(X)$ of all complex-valued continuous functions on maximal ideal space X of A. A self-adjoint element $a \in A$ is called positive if $\sigma(a) \subset \{r \in \mathbb{R} : r \geq 0\}$. The set of all positive elements in A is denoted by A_+. A matrix in $M_n(\mathbb{C})$ is positive if and only if it is a positive semidefinite matrix.

In this note we exhibit the commutativity of C^*-algebras and pairs of self-adjoint elements according to [1]. We begin with a classical theorem of Jacobson.

- Let R be a ring. Jacobson proved the following: Suppose that $\forall x \in R\ \exists n(x) > 1$ with $x^{n(x)} = x$. Then R is commutative.
- N. Herstein give a generalization of a theorem of Jacobson (Amer. J. Math. 73 (1951), 756–762).

After that several conditions for a C^*-algebra to be commutative are studied.

- Sherman (Amer. J. Math.,1951), A_+ is a lattice,
- Ogasawara (J. Sci. Hiroshima Univ.,1955),$a \geq b$ always implies $a^2 \geq b^2$,
- Fukamia and Misonou and Takeda (Tohoku Math. J.,1954), A has decomposition property,
- Nakamoto (Math. Japon., 1979/1980), in terms of spectrum of elements,
- Wu (Proc. AMS, 2001), order characterization of commutativity, $e^{x+y} = e^x e^y$ for all x, y ,
- Ji and Tomiyama (Proc. AMS, 2003) : existence of continuous monotone scalar function on the positive axis which is not matrix monotone of order 2 but operatormnotone on A_+. A local characterization. $\exp x$ is two positive.

Algebraic character are the following:

- Kaplansky (Dixmier's book, 1969), 0 is the only nilpotent element,
- Jeang and Ko (Manuscripta Math., 2004): properties on the functional calculus,
- Beneduci and Molnár (Jour. Math. Anal. Appl., 2014), K-loop properties
- Molnár (Abstr. Appl. Anal. 2014), algebraic properties of power functions, the logarithmic and exponential functions, and the sine and cosine functions.

2 Theorems of Jeang and Ko

The following is a theorem of Jeang and Ko [2].

Theorem 1 (Jeang and Ko)**.** *Let A be a unital C^*-algebra. Let $f, g : I \to \mathbb{C}$ be a non-constant complex-valued continuous functions on a degenerate interval. Suppose that*

$$f(x)g(y) = g(y)f(x), \quad x, y \in A_{sa} \text{ with } \sigma(x), \sigma(y) \subset I.$$

Then we have that A is commutative.

We look at shortly the way of Jeang and Ko.

Definition 2.1. *Let A be a unital C^*-algebra. Suppose that $f : I \to \mathbb{C}$ is a continuous function defined on a non-degenerate interval. Let*
$f(A_{sa})$ = the complex linear span of $\{f(x) : x \in A_{sa}, \sigma(x) \subset I\}$.
We say that

(1) f densely spans A if $f(A_{sa})$ is dense in A.

(2) f totally spans A if $f(A_{sa}) = A$.

Jeang and Ko [2] in fact proved that

Theorem 2 (Jeang and Ko)**.** *Let A be a unital C^*-algebra. Suppose that f is a non-constant complex-valued function defined on a non-degenerate interval. Then f densely spans A, i.e., $\overline{f(A_{sa})} = A$*

As a corollary

Theorem 3 (Jeang and Ko)**.** *Let A be a unital C^*-algebra. Suppose that $f, g : I \to \mathbb{C}$ are non-constant continuous functions defined on a non-degenerate interval I. Suppose that*

$$f(x)g(y) = g(y)f(x), \quad x, y \in A_{sa} \text{ with } \sigma(x), \sigma(y) \subset I.$$

Then A is commutative.

The purpose of this note is to exhibit the following according to [1]:

(1) a local version of Theorem (Jeang and Ko),

(2) a negative answer to the problem posed by Jeang and Ko [2].

Simply a local version of a theorem of Jeang and Ko might be the following.

Question 4. $f, g : I \to \mathbb{C}$, *continuous, non-constant*
If $x, y \in A_{sa}$ satisfies

$$f(x)g(y) = g(y)f(x)$$

$\overset{?}{\Longrightarrow}$

$$xy = yx$$

Sometimes it is true and sometimes it is false.

- True for $f(t) = g(t) = \exp(t)$.

  Suppose that $\exp x \exp y = \exp y \exp x$ for $x, y \in A_{sa}$. Then $p(\exp x)p(\exp y) = p(\exp y)p(\exp x)$ for any polynomial with real coefficients. By the Weierstrass approximation theorem for continuous functions, there exists a sequence of polynomials which uniformly approximate $\log \cdot$ on a compact set which include the union of the spectrum of x and y. Hence we have $xy = yx$.

- False for $f(t) = g(t) = \exp(it)$.

  If
  $$A = \begin{pmatrix} 0 & 2\pi i \\ -2\pi i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
  then $e^{iA} = E$, hence $e^{iA}e^{iB} = e^{iB}e^{iA}$, while $AB \neq BA$.

- It is easy to have a similar example for $f(t) = g(t) = \sin t, \cos t$; false for $f(t) = g(t) = \sin t$ or $\cos t$.

- The Cantor ternary function is constant on an interval. Hence it is false for the Cantor ternary function.

3 Localization of theorems of Jeang and Ko

In this section we exhibit a local version of theorems of Jeang and Ko. The following is proved in [1].

Theorem 5. *Let A be a unital C^*-algebra. Suppose that $f : I \to \mathbb{C}$ is a non-constant complex-valued continuous function on a non-degenerate interval I. Let $a \in A_{sa}$. Then a is in the closed linear span of*
$$\{f(t + sa) : t, s \in \mathbb{R}, \sigma(t + sa) \subset I\}.$$

Proof. Put $L = \{f(t + sa) : t, s \in \mathbb{R}, \sigma(t + sa) \subset I\}$. Suppose first that f is continuously differentiable. Then
$$(f(t_0 + sa) - f(t_0))/s \to f'(t_0)a \in L.$$
We can choose $f'(t_0) \neq 0$. Hence $a \in L$.

Next we consider the general case. Without loss of generality we may assume that $I = \mathbb{R}$. Let $w : \mathbb{R} \to \mathbb{R}$ be a non-negative continuously differentiable function whose support is contained in $(-\delta, \delta)$ for a small $\delta > 0$ such that $\int_{-\delta}^{\delta} w(s)ds = 1$. Put

$$f_w(t) = \int_{-\delta}^{\delta} f(t - y)w(y)dy, \quad t \in [\delta, 1 - \delta]. \tag{1}$$

Then f_w is continuously differentiable, and f_w is non-constant for a sufficiently small δ. As $f_w(a) \in L$, we have $f'_w(t_0)a \in L$, where $f'_w(t_0) \neq 0$. Hence $a \in L$. □

By the above theorem we have the following (cf. [1]).

Corollary 6. *Let A be a unital C^*-algebra. Let $f, g : I \to \mathbb{C}$ be a non-constant complex-valued continuous function on a non-degenerate interval I. Let $x, y \in A_{sa}$. Suppose that*

$$f(t+sx)g(t'+s'y) = g(t'+s'y)f(t+sx)$$

for every quarter $t, s, t', s' \in \mathbb{R}$ with$\sigma(t+sx), \sigma(t'+s'y) \subset I$. Then we have $xy = yx$

4 A problem of Jeang and Ko and a negative answer

Recall that $f(A_{sa})$ = the complex linear span of $\{f(a) : a \in A_{sa}, \sigma(a) \subset I\}$. Jeang and Ko proposed the following problem in [2].

Problem 7. *Does any non-constant continuous function f totally span A?*

$$f(A_{sa}) = A?$$

They give a partial answer to the problem in the sense that it is the case if f is *strictly monotone* [2]. The following is a negative answer to the problem. A precise proof is given in [1].

Example ([1])**.** *For the Cantor ternary function φ*

$$1 \notin \varphi(C([0,1])_{sa}).$$

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Commutativity for C^*-algebras via gyrogroup operations

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1 Introduction

Let A be a unital C^*-algebra. Recall that an element $a \in A$ is positive iff $a = a^*$ and $\sigma(a) \subset \{r \in \mathbb{R} : r \geq 0\}$ iff $a = b^*b$ for some $b \in A$. In this note we denote by A_+^{-1} the set of all positive invertible elements in A. The following is well known.

- $C(X)_+^{-1} = \{f \in C(X) : f > 0\}$, where $C(X)$ denotes the commutative C^*-algebra of all complex-valued continuous functions on a compact Hausdorff space X.
- Unless A is commutative, ab need not be positive in general for a pair of positive elements a and b.
- A_+^{-1} need not be an group.
- A_+^{-1} is a twisted subgroup of A^{-1}, i.e., $ab^{-1}a \in A_+^{-1}$ for every $a, b \in A_+^{-1}$, where A^{-1} is the general linear group of A.

In this note we exhibit the commutativity of positive elements in a unital C^*-algebra according to [1].

Fact 1. *Suppose that a and b are positive. Then ab is positive iff $ab = ba$. Hence A is commutative iff A_+^{-1} is a group.*

Proof. If ab is positive, then $ab = (ab)^* = b^*a^* = ba$. Conversely if $ab = ba$, then $a^{\frac{1}{2}}$ and b commute since $a^{\frac{1}{2}}$ is approximated by polynomials of a. Thus $ab = a^{\frac{1}{2}}ba^{\frac{1}{2}}$ is positive. □

Although A_+^{-1} is a twisted subgroup of A^{-1}, A_+^{-1} need not be a subgroup of A^{-1}.

2 Gyrocommutative Gyrogroups

In fact A_+^{-1} is a gyrocommutative gyrogroup.

Definition 2.1. *A groupoid $(G, \oplus)$ is a gyrogroup if there exists a point $\epsilon \in G$ such that the following hold.*

(G1) $\forall \boldsymbol{a} \in G\ \epsilon \oplus \boldsymbol{a} = \boldsymbol{a}$, .

(G2) $\forall \boldsymbol{a} \in G\ \ \exists \ominus \boldsymbol{a}$ *s.t.* $\ominus \boldsymbol{a} \oplus \boldsymbol{a} = \epsilon$.

(G3) $\forall \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in G\ \ \exists!\, \mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c} \in G$ *s.t.* $\boldsymbol{a} \oplus (\boldsymbol{b} \oplus \boldsymbol{c}) = (\boldsymbol{a} \oplus \boldsymbol{b}) \oplus \mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c}$.

(G4) $\mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}]$ *is an gyroautomorphism for* $\forall \boldsymbol{a}, \boldsymbol{b} \in G$

(G5) $\forall \boldsymbol{a}, \boldsymbol{b} \in G\ \mathrm{gyr}[\boldsymbol{a} \oplus \boldsymbol{b}, \boldsymbol{b}] = \mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}]$.

Gyrocommutative if the following (G6) is also satisfied.

(G6) $\forall \boldsymbol{a}, \boldsymbol{b} \in G\ \boldsymbol{a} \oplus \boldsymbol{b} = \mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}](\boldsymbol{b} \oplus \boldsymbol{a})$.

The following is exhibited in [2].

Theorem 1 (A_+^{-1} is a gyrocommutative gyrogroup)**.** *For* $0 < t \in \mathbb{R}$*, put*

$$a \oplus_t b = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}}, \quad a, b \in A_+^{-1}$$

Then $(A_+^{-1}, \oplus_t)$ *is a gyrocommutative gyrogroup. The gyrogroup identity is the identity element* e *of* A *as* C^**-algebra. The inverse element* $\ominus a$ *is* a^{-1} *For* $a, b \in A_+^{-1}$ *put*

$$X = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} a^{\frac{t}{2}} b^{\frac{t}{2}}.$$

Then X *is the unitary part of the polar decomposition of* $a^{\frac{t}{2}} b^{\frac{t}{2}}$ *and*

$$\mathrm{gyr}_t[a, b]c = XcX^*, \quad a, b, c \in A_+^{-1}.$$

Note that Beneduci and Molnár [3] showed that $(A_+^{-1}, \oplus_1)$ is a K-loop, which is equivalent to a gyrocommutative gyrogroup.

3 Commutativity via gyrogroup operation

Beneduci and Molnár [3] proved that $ab = ba$ if and only if $a \oplus_1 b = b \oplus_1 a$. The following is proved in [1].

Theorem 2. *Let* $a, b \in A_+^{-1}$*. The following are equivalent*

(1) $ab = ba$*,*

(2) ab *is self-adjoint,*

(3) ab *is positive,*

(3)' *the unitary part of* ab *is in the center of* A*,*

(4) $\exists(\forall) t > 0$ *such that* $\mathrm{gyr}_t[a, b]$ *is the identity map on* A_+^{-1}*,*

(5) $\exists(\forall) t > 0$ *such that* $(a \oplus_t b) \oplus_t c = a \oplus_t (b \oplus_t c)$ *for every* $c \in A_+^{-1}$*,*

(6) $\exists(\forall) t > 0$ *such that* $a \oplus_t b = b \oplus_t a$*,*

(7) $\exists(\forall)t > 0$ *such that* $ab = a \oplus_t b$.

Proof. We give a sketch proof. By an easy calculation we have (1), (2) and (3) are equivalent.

It is trivial that (3); ab is positive, is followed by (3)' the unitary part of ab is in the center of A.

Assume (3)'; the unitary part of ab is in the center. As $\mathrm{gyr}_2[a,b]$ is the unitary transformation by the unitary part of ab, we infer that $\mathrm{gyr}_2[a,b]$ is trivial; (4) for $t = 2$ holds; $\mathrm{gyr}_2[a,b]$ is the identity map on A_+^{-1}.

Assume (4) for $t = 2$; $\mathrm{gyr}_2[a,b]$ is trivial. Then (5) for $t = 2$; $(a \oplus_2 b) \oplus_2 c = a \oplus_2 (b \oplus_2 c)$ for every $c \in A_+^{-1}$, is trivial by $(a \oplus_2 b) \oplus_2 \mathrm{gyr}_2[a,b]c = a \oplus_2 (b \oplus_2 c)$.

Assume (5) for $t = 2$. Then $\mathrm{gyr}_2[a,b]$ is the identity since

$$(a \oplus_2 b) \oplus_2 \mathrm{gyr}_2[a,b]c = a \oplus_2 (b \oplus_2 c) = (a \oplus_2 b) \oplus_2 c$$

for every $c \in A_+^{-1}$. Thus (6) for $t = 2$ holds; $a \oplus_2 b = b \oplus_2 a$.

Suppose that (6) ;

$$(a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}} = (b^{\frac{t}{2}} a^t b^{\frac{t}{2}})^{\frac{1}{t}}.$$

Then,

$$(a^{\frac{t}{2}} b^{\frac{t}{2}})(b^{\frac{t}{2}} a^{\frac{t}{2}}) = (b^{\frac{t}{2}} a^{\frac{t}{2}})(a^{\frac{t}{2}} b^{\frac{t}{2}})$$

and

$$(a^{\frac{t}{2}} b^{\frac{t}{2}})(a^{\frac{t}{2}} b^{\frac{t}{2}})^* = (a^{\frac{t}{2}} b^{\frac{t}{2}})^*(a^{\frac{t}{2}} b^{\frac{t}{2}}).$$

Hence $(a^{\frac{t}{2}} b^{\frac{t}{2}})$ is normal. As $\sigma(a^{\frac{t}{2}} b^{\frac{t}{2}}) = \sigma(a^{\frac{t}{4}} b^{\frac{t}{2}} a^{\frac{t}{4}}) \geq 0$, $a^{\frac{t}{2}} b^{\frac{t}{2}}$ is is self-adjoint. We have

$$a^{\frac{t}{2}} b^{\frac{t}{2}} = (a^{\frac{t}{2}} b^{\frac{t}{2}})^* = b^{\frac{t}{2}} a^{\frac{t}{2}}.$$

As a (resp. b) is approximated by a polynomial of $a^{\frac{t}{2}}$ (resp. $b^{\frac{t}{2}}$) we have $ab = ba$. Then (1) holds.

The implications (1)→(7) and (7)→(3) are trivial. □

Applying Theorem 2 to get

Corollary 3. *The following are equivalent.*

(1) A *is commutative,*

(2) $\exists(\forall)t > 0$ *such that* $\mathrm{gyr}_t[a,b]$ *is the identity map on* A_+^{-1} *for* $\forall a, b \in A_+^{-1}$,

(3) $\exists(\forall)t > 0$ *such that* $(a \oplus_t b) \oplus_t c = a \oplus_t (b \oplus_t c)$ *for* $\forall a, b, c \in A_+^{-1}$,

(4) $\exists(\forall)t > 0$ *such that* $a \oplus_t b = b \oplus_t a$ *for* $\forall a, b \in A_+^{-1}$,

(5) $\exists(\forall)t > 0$ *such that* $ab = a \oplus_t b$ *for* $\forall a, b \in A_+^{-1}$,

(6) $\exists(\forall)t > 0$ *such that* $(A_+^{-1}, \oplus_t)$ *is a group,*

(7) $\exists(\forall)t > 0$ *such that* $(A_+^{-1}, \oplus_t)$ *is a commutative group.*

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Related subjects of Invariant Subspace Problem in the Hardy space

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[1] The Hardy space H^2 on $\mathbb{D}$. $\theta \in H^2$ is called inner if $|\theta| = 1$ a.e. on $\mathbb{T}$. For $\alpha \in \mathbb{D}$, write $b_\alpha(z) = (z - \alpha)/(1 - \overline{\alpha} z)$. For distinct points $\{\alpha_n\}_n$ in $\mathbb{D}$ and positive integers $\{k_n\}_n$ satisfying $\sum_{n=1}^\infty k_n(1 - |\alpha_n|) < \infty$, define

$$b(z) = \prod_{n=1}^\infty \Big(\frac{-\overline{\alpha}_n}{|\alpha_n|} \, b_{\alpha_n}(z) \Big)^{k_n}, \quad \text{a Blaschke product.}$$

For a bounded positive singular measure μ on $\mathbb{T}$, define

$$\psi_\mu(z) = \exp\Big(- \int_{\mathbb{T}} \frac{e^{it} + z}{e^{it} - z} \, \mu(e^{it}) \Big), \quad \text{a singular inner function.}$$

For $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator is defined by $T_\varphi f = P(\varphi f)$ for $f \in H^2$. We have $T_\varphi^* = T_{\overline{\varphi}}$, and T_z, T_z^* are called the forward, backward shifts, respectively. A closed subspace M of H^2 is called invariant if $T_z M \subset M$. When $M \neq \{0\}$, by the Beurling theorem $M = \theta H^2$ for an inner function θ. A closed subspace N of H^2 is called backward shift invariant if $T_z^* N \subset N$. In this case, $H^2 \ominus N$ is an invariant subspace. When $N \neq H^2$, $N = H^2 \ominus \theta H^2$ for some inner function θ.

For $f \in H^2$ and $M \subset H^2$, write $[f]_*$ and $[M]_*$ the smallest backward shift invariant subspaces of H^2 containing f and M, respectively.

[2] Invariant subspace Problem. Let H be a separable Hilbert space. Write $B(H)$ the set of bounded linear operators on H. A closed subspace $M \subset H$ is called invariant for $T \in B(H)$ if $TM \subset M$.

Invariant subspace Problem (ISP): For every $T \in B(H)$, is there a proper invariant subspace M for T (a non-cyclic vector in H for T)?

An operator $T \in B(H)$ is called universal if “undefined here”. By the definition of the universality, ISP is equivalent to the following.

Problem A. Let $T \in B(H)$ be universal. For every invariant subspace M for T with $\dim M = \infty$, is there an invariant subspace M_0 for T satisfying $\{0\} \subsetneqq M_0 \subsetneqq M$?

Fact 1 (Caradus, 1969). For $T \in B(H)$, if $\dim \ker T = \infty$ and $TH = H$, then T is universal.

Let $\mathcal{U} = \{T_\varphi^* : \varphi \in H^\infty, 1/\varphi \in L^\infty(\mathbb{T}), \dim \ker T_\varphi^* = \infty\}$. Then by Fact 1, it is not difficult to see that $T_\varphi^* \in \mathcal{U}$ is universal.

Fact 2. Let θ be an inner function. If θ is not a finite Blaschke product, then T_θ^* is universal.

Then ISP is equivalent to the following.

Problem B. Let θ be inner but not a finite Blaschke product. For every invariant subspace $M \subset H^2$ for T_θ^* with $\dim M = \infty$, is there an invariant subspace M_0 for T_θ^* satisfying $\{0\} \subsetneqq M_0 \subsetneqq M$?

Fact 3. Let θ be inner and $M \subset H^2$ a closed subspace. If $T_z^* M \subset M$, then $T_\theta^* M \subset M$.

Cowen-Gallardo's strategy. Let θ be inner but not a finite Blaschke product and $M \subset H^2$ an invariant subspace for T_θ^* with $\dim M = \infty$. Find a nonzero $f \in M$ such that $M \not\subset [f]_*$. Then $f \in M \cap [f]_* \subsetneqq M$ and $T_\theta^*(M \cap [f]_*) \subset M \cap [f]_*$ (by Fact 3). Hence ISP is solved!

Difficulty. For an inner function θ which is not a finite Blaschke product, it is difficult to describe all invariant subspaces for T_θ^*.

Today's subject. Let θ be inner but not a finite Blaschke product. For which closed subspaces $M \subset H^2$ with $\dim M = \infty$, is there a nonzero $f \in M$ such that $M \not\subset [f]_*$?

[3] **Cowen-Gallardo's questions.** Here θ denotes inner but not a finite Blaschke product. There are many $f \in H^2$ satisfying $[f]_* = H^2$.

Question 0. Does every closed subspace $M \subset H^2$ with $\dim M = \infty$ include a nonzero f such that $[f]_* \neq H^2$ (f is non-cyclic for T_z^*)?

Nikolski's answer: NO. Let $S = \{n_j\}_j$ be a sequence of positive integers such that $\inf_{j>k} n_j/n_k > 1$ (a lacunary sequence). Let $f = \sum_{n \in S} \hat{f}(n) z^n \in H^2$ and $\hat{f}(n) \neq 0$ for every $n \in S$. It is known that $[f]_* = H^2$. Let $S = \bigcup_{k=1}^\infty S_k$ be a union of disjoint infinite sets. For each $k \geq 1$, let $f_k = \sum_{n \in S_k} \hat{f}(n) z^n \in H^2$. We have $f_k \perp f_j$ for $k \neq j$, and set $M = \bigoplus_{k=1}^\infty \mathbb{C} \cdot f_k$. Then every nonzero $g \in M$, $\{n : \hat{g}(n) \neq 0\}$ is a lacunary sequence, so $[g]_* = H^2$. □

Q1. Does every closed subspace $M \subset H^2$ with $[M]_* = H^2$ include f such that $[f]_* = H^2$?

Q2. Does every closed subspace $M \subset H^2 \ominus \theta H^2$ with $[M]_* = H^2 \ominus \theta H^2$ include f such that $[f]_* = H^2 \ominus \theta H^2$? (See Question 3.)

The following three questions are posed by Cowen and Gallardo [1].

Question 1. Does every closed subspace $M \subset H^2$ with $\dim M = \infty$ that is a proper (?) invariant subspace for an operator in $\mathcal{U}$ include a nonzero f such that $[f]_* \neq H^2$?

Question 2. Let $M \subset H^2$ be a closed subspace with $\dim M = \infty$ such that $TM \subset M$ for some $T \in \mathcal{U}$ and θ be inner satisfying $M \subset H^2 \ominus \theta H^2$ (we may assume $[M]_* = H^2 \ominus \theta H^2$). Is there always $f \in M$ such that $\{0\} \not\subset [f]_* \subsetneqq M$?

Question 3. Is there a closed subspace $M \subset H^2$ with $\dim M = \infty$ and $[M]_* = H^2 \ominus \theta H^2$ for some inner θ such that $[f]_* = H^2 \ominus \theta H^2$ for every nonzero $f \in M$?

The following example answers to Question 3 affirmatively.

Example 1. Consider that $\theta = \psi_{\delta_1}$. Take $\{t_n\}_{n\geq 1}$ such that $0 < t_n < t_{n+1}$ and $t_n \to 1$. Set $E_n = \psi_{t_n\delta_1} H^2 \ominus \psi_{t_{n+1}\delta_1} H^2$ and $\xi_n = \psi_{t_{n+1}\delta_1}/\psi_{t_n\delta_1} = \psi_{(t_{n+1}-t_n)\delta_1}$. Then

$$H^2 \ominus \psi_{\delta_1} H^2 \supset \psi_{t_1\delta_1} H^2 \ominus \psi_{\delta_1} H^2 = \bigoplus_{n=1}^{\infty} \psi_{t_n\delta_1} H^2 \ominus \psi_{t_{n+1}\delta_1} H^2$$

$$= \bigoplus_{n=1}^{\infty} \psi_{t_n\delta_1}(H^2 \ominus \xi_n H^2).$$

Take $f_n \in H^2 \ominus \xi_n H^2$ with $\|f_n\| = 1$. We have $\psi_{t_n\delta_1} f_n \perp \psi_{t_i\delta_1} f_i$ for $n \neq i$. Take $\{c_n\}_{n\geq 1}$ in $\mathbb{C}$ such that $\sum_{n=1}^{\infty} |c_n|^2 < \infty$ and $c_n \neq 0$. Then

$$\sum_{n=1}^{\infty} c_n \psi_{t_n\delta_1} f_n \in H^2 \ominus \psi_{\delta_1} H^2.$$

Take a sequence of mutually disjoint sets of positive integers $\{N_k\}_{k\geq 1}$ such that N_k is an infinite set. For each $k \geq 1$, let

$$F_k = \sum_{n \in N_k} c_n \psi_{t_n\delta_1} f_n \in H^2 \ominus \psi_{\delta_1} H^2.$$

We have $F_k \perp F_i$ for $k \neq i$, and let $M = \bigoplus_{k=1}^{\infty} \mathbb{C} \cdot F_k$. Then $M \subset H^2 \ominus \psi_{\delta_1} H^2$, M is closed and $\dim M = \infty$.

It is not difficult to see that for $f \in H^2 \ominus \psi_{\delta_1} H^2$, $[f]_* = H^2 \ominus \psi_{\delta_1} H^2$ if and only if $f \not\perp \psi_{t\delta_1} H^2$ for every $0 < t < 1$. Then $[F]_* = H^2 \ominus \psi_{\delta_1} H^2$ for every nonzero $F \in M$. $\square$

Remark 1. Example 1 also answers to Question 2 negatively in some sense. Let M be given in Example 1 and $T = T^*_{\psi_{\delta_1}} \in \mathcal{U}$. Since $M \subset H^2 \ominus \psi_{\delta_1} H^2$, $T^*_{\psi_{\delta_1}} = 0$ on M, so $T^*_{\psi_{\delta_1}} M \subset M$. By Example 1, $M \subsetneqq H^2 \ominus \psi_{\delta_1} H^2 = [f]_*$ for every nonzero $f \in M$.

For $\alpha \in \mathbb{D}$ with $\alpha \neq 0$, let $\varphi = (\psi_{\delta_1} - \alpha)/(1 - \overline{\alpha}\psi_{\delta_1})$. It is known that φ is a Blaschke product. We also have $T_\varphi^* = \overline{\alpha} I$ on M, so $T_\varphi^* M \subset M$. Note that $T_\varphi^*(\mathbb{C} \cdot F) \subset \mathbb{C} \cdot F$ for every $F \in M$. I no not know what is real Question 2? □

Q3. Let μ be a positive singular measure on $\mathbb{T}$. Is there a closed subspace $M \subset H^2 \ominus \psi_\mu H^2$ with $\dim M = \infty$ and $[M]_* = H^2 \ominus \psi_\mu H^2$ such that $[f]_* = H^2 \ominus \psi_\mu H^2$ for every nonzero $f \in M$?

Next, we shall study Q2.

Proposition 1. Suppose that $\theta(\alpha) = 0$ for some $\alpha \in \mathbb{D}$. Let M be a closed subspace of $H^2 \ominus \theta H^2$ such that $[M]_* = H^2 \ominus \theta H^2$ and $\dim M \geq 2$. Then there exists a nonzero $f_0 \in M$ satisfying that $[f]_* \neq H^2 \ominus \theta H^2$ for every $f \in M$ with $f \perp f_0$.

Proof. Write $\theta = b_\alpha \theta_1$ for an inner function θ_1. Then

$$M \not\perp \frac{\theta_1}{1 - \overline{\alpha} z} \in H^2 \ominus \theta H^2.$$

Put $f_0 = P_M\left(\frac{\theta_1}{1-\overline{\alpha}z}\right) \neq 0$. We have the assertion. □

In Proposition 1, both cases occur; $[f_0]_* = H^2 \ominus \theta H^2$ and $[f_0]_* \subsetneqq H^2 \ominus \theta H^2$.

Example 2. Let θ_1 be a singular inner function and $\theta = z\theta_1$. Let

$$M := [\theta_1]_* = H^2 \ominus z\theta_1 H^2 = H^2 \ominus \theta H^2.$$

We have $\theta(0) = 0$ and $M = (H^2 \ominus \theta_1 H^2) \oplus \mathbb{C} \cdot \theta_1$. Let $f_0 \in M$ be given in Proposition 1. We have $f_0 = P_M \theta_1 = \theta_1$, so $[f_0]_* = [\theta_1]_* = H^2 \ominus \theta H^2$.

Since $1 \in M$ and $1 \notin H^2 \ominus \theta_1 H^2$, $M = (H^2 \ominus \theta_1 H^2) + \mathbb{C} \cdot 1$. Let

$$M_1 = \{f \in H^2 \ominus \theta_1 H^2 : f \perp 1\} \oplus \mathbb{C} \cdot 1.$$

It is not difficult to see that

$$H^2 \ominus \theta_1 H^2 = \big[\{f \in H^2 \ominus \theta_1 H^2 : f \perp 1\}\big]_*.$$

Then $[M_1]_* = H^2 \ominus \theta H^2$. Let $f_1 = P_{M_1}\theta_1$. We have $f_1 = \langle P_{M_1}\theta_1, 1\rangle 1 = \theta_1(0)1$ and $[f_1]_* = H^2 \ominus zH^2 \neq H^2 \ominus \theta H^2$.

We shall show the existence of $f_3 \in M_1$ satisfying $[f_3]_* = H^2 \ominus \theta H^2$. There is singular inner θ_2 such that $\theta_2^2 = \theta_1$. Then

$$1 - \overline{\theta_2(0)}\theta_2, \;\; \theta_2(1 - \overline{\theta_2(0)}\theta_2) \in H^2 \ominus \theta_1 H^2.$$

We have

$$f_2 := \theta_2(1 - \overline{\theta_2(0)}\theta_2) - \theta_2(0)(1 - \overline{\theta_2(0)}\theta_2) \in \{f \in H^2 \ominus \theta_1 H^2 : f \perp 1\}$$

and $f_3 := f_2 + 1 \in M_1$. Since $T^*_{\theta_1} f_3 = T^*_{\theta_1} 1 = \overline{\theta_1(0)} 1$, $\mathbb{C} \cdot 1 \subset [f_3]_*$. We also have $T^*_{z\theta_2} f_3 = -\overline{\theta_2(0)} T^*_z \theta_2$, so $T^*_z \theta_2 \in [f_3]_*$. Since $[T^*_z \theta_2]_* = H^2 \ominus \theta_2 H^2$, $H^2 \ominus \theta_2 H^2 \subset [f_3]_*$. Hence $\theta_2(1 - \overline{\theta_2(0)}\theta_2) \in [f_3]_*$. Therefore

$$\begin{aligned} H^2 \ominus \theta_1 H^2 &= (H^2 \ominus \theta_2 H^2) \oplus \theta_2 (H^2 \ominus \theta_2 H^2) \\ &\subset (H^2 \ominus \theta_2 H^2) + \big[\theta_2 (1 - \overline{\theta_2(0)} \theta_2)\big]_* \subset [f_3]_*. \end{aligned}$$

Thus $H^2 \ominus \theta H^2 = (H^2 \ominus \theta_1 H^2) + \mathbb{C} \cdot 1 \subset [f_3]_* \subset H^2 \ominus \theta H^2$. □

Proposition 2. Let θ be a Blaschke product and M be a closed subspace of $H^2 \ominus \theta H^2$ such that $[M]_* = H^2 \ominus \theta H^2$. Then there is $f \in M$ satisfying $[f]_* = H^2 \ominus \theta H^2$.

Proof. Write

$$\theta = \prod_{n=1}^{\infty} \Big(\frac{-\overline{\alpha}_n}{|\alpha_n|} b_{\alpha_n} \Big)^{k_n}.$$

For $j \geq 1$, let

$$\theta_j = \frac{\theta}{\frac{-\overline{\alpha}_j}{|\alpha_j|} b_{\alpha_j}}.$$

Then

$$H^2 \ominus \theta H^2 = (H^2 \ominus \theta_j H^2) \oplus \mathbb{C} \cdot \frac{\theta_j}{1 - \overline{\alpha}_j z}.$$

Since $[M]_* = H^2 \ominus \theta H^2$, $f \not\perp \frac{\theta_j}{1 - \overline{\alpha}_j z}$ for some $f \in M$. Let

$$M_j := \Big\{ f \in M : f \perp \frac{\theta_j}{1 - \overline{\alpha}_j z} \Big\}.$$

Then M_j is a closed subspace of M and does not contains a non-void open subset of M. If $\bigcup_{j=1}^{\infty} M_j \subsetneqq M$, then there is $f \in M$ such that $f \not\perp \theta_j / (1 - \overline{\alpha}_j z)$ for every $j \geq 1$. In this case, it is not difficult to see the assertion. If $\bigcup_{j=1}^{\infty} M_j = M$, then by the Baire category theorem there is $j_0 \geq 1$ such that M_{j_0} contains a non-void open subset of M. This is a contradiction. □

Remark 2. When θ is singular inner associated with a discrete positive measure, Proposition 2 also holds (in the same way). For, write $\theta = \psi_\mu$, where $\mu = \sum_j c_j \delta_{\lambda_j}$. For $\{t_n\}_n$ satisfying $0 < t_n < t_{n+1} \to 1$, use a countable set

$$\Big\{ \Big(\prod_{j=1, j \neq k}^{\infty} \psi_{c_j \delta_{\lambda_j}} \Big) T^*_z \psi_{c_k t_n \delta_{\lambda_k}} \Big\}_{k,n}.$$

Q4. Let θ be singular inner associated with a continuous positive measure and M be a closed subspace of $H^2 \ominus \theta H^2$ such that $[M]_* = H^2 \ominus \theta H^2$. Is there $f \in M$ satisfying $[f]_* = H^2 \ominus \theta H^2$?

[4] Another example for Question 0. Let θ be a non-constant inner function and g a nonzero function in H^2. Write $g = \sum_{n=0}^{\infty} a_n z^n$. For $f \in H^2 \ominus \theta H^2$, we define

$$\Phi(f, g) = \sum_{n=0}^{\infty} a_n f \theta^n.$$

Since $f\theta^n \perp f\theta^m$ for $n \neq m$, we have

$$\|\Phi(f, g)\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \|f\|^2 = \|f\|^2 \|g\|^2.$$

Then $\Phi : (H^2 \ominus \theta H^2) \times H^2 \to H^2$ is a separately bounded linear operator and bounded below. We have

$$T_\theta^* \Phi(f, g) = \sum_{n=1}^{\infty} a_n f \theta^{n-1} = \Phi(f, T_z^* g) \tag{1}$$

and for $f_1 \in H^2 \ominus \theta H^2$, $\langle \Phi(f, g), \Phi(f_1, g)\rangle = \langle f, f_1\rangle \|g\|^2$.

Proposition 3. If $[f]_* = H^2 \ominus \theta H^2$ and $[g]_* = H^2$, then $[\Phi(f, g)]_* = H^2$.

Proof. Since $[g]_* = H^2$, for $k \geq 0$ there is a sequence of polynomials $\{p_n\}_{n\geq 1}$ such that $T_{p_n}^* g \to z^k$ as $n \to \infty$. By (1),

$$T_{p_n \circ \theta}^* \Phi(f, g) = \Phi(f, T_{p_n}^* g) \to \Phi(f, z^k) = f\theta^k.$$

Then $f\theta^k \in [\Phi(f, g)]_*$ for $k \geq 0$, so $f \in [\Phi(f, g)]_*$. Since $[f]_* = H^2 \ominus \theta H^2$, we have $H^2 \ominus \theta H^2 \subset [\Phi(f, g)]_*$. Since

$$T_z^*(f\theta) = (T_z^* f)\theta + f(0) T_z^* \theta,$$

we have $(T_z^* f)\theta \in [\Phi(f, g)]_*$. Repeating the same argument, $(T_z^{*j} f)\theta \in [\Phi(f, g)]_*$ for every $j \geq 0$. Hence $(H^2 \ominus \theta H^2)\theta \subset [\Phi(f, g)]_*$. Repeatedly we have $(H^2 \ominus \theta H^2)\theta^j \subset [\Phi(f, g)]_*$ for every $j \geq 0$. Hence we have $H^2 = \bigoplus_{j=0}^{\infty} (H^2 \ominus \theta H^2)\theta^j \subset [\Phi(f, g)]_*$. Thus we get the assertion. □

Example 3. Let $g \in H^2$ satisfy $[g]_* = H^2$ and $\theta = \psi_{\delta_1}$. Let M be a closed subspace given in Example 1. Then by Proposition 3, $\Phi(M, g) := \{\Phi(f, g) : f \in M\}$ is closed, $\dim \Phi(M, g) = \infty$, and $[\Phi(f, g)]_* = H^2$ for every $f \in M$. □

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CYCLICITY OF REPRODUCING KERNELS IN WEIGHTED HARDYBSPACES OVER THE BIDISK

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1. Introduction

複素空間 $\mathbb{C}^n$ の領域を Ω とし、$Hol(\Omega)$ を Ω 上の正則関数全体、$\mathcal{C}^n$ を $\mathbb{C}^n$ 上の多項式環とする。$\mathcal{H} \subset Hol(\Omega)$ を Hilbert 空間とする。

定義 1. M を $\mathcal{H}$ の閉部分空間とする。そのとき、任意の多項式 $p \in \mathcal{C}^n$ に対して、$pM \subset M$ が成り立つならば M は不変であるという。

さらに、$\mathcal{H}$ の部分集合 E に対して次のように表記する：

- $[E]_{\mathcal{H}}$ によって、E を含む $\mathcal{H}$ の最小の不変部分空間を表す、
- $[E]_{\mathcal{H}} = M$ ならば、E を M の生成集合と呼ぶ、
- M の生成集合の要素の最小個数をランクと呼び、$rank_{\mathcal{H}} M$ によって表す。

1949 年に Beurling によって、Hardy 空間 $H^2(\mathbb{D})$ の不変部分空間の特徴付けが行われた。

Beurling の定理 (1949). M を $H^2(\mathbb{D})$ の閉部分空間とする。そのとき、M が不変であることと、$M = \varphi(z)H^2(\mathbb{D})$, $\varphi(z)$: 内部関数、であることは同値である。

この定理より、$M \ominus zM = \mathbb{C} \cdot \varphi$ がわかり、

$$[M \ominus zM]_{H^2(\mathbb{D})} = [\varphi]_{H^2(\mathbb{D})} = M \quad and \quad rank_{H^2(\mathbb{D})} M = 1$$

を満たす。この $[M \ominus zM]_{\mathcal{H}} = M$ を Beurling property という。以上のように、$H^2(\mathbb{D})$ の任意の不変部分空間が Beurling property を持つが、1 変数 Bergman 空間 [1] や Dirichlet 空間 [6] においても同様であることが知られている。

では、多変数解析関数空間ではどうであるか。いま 2 次複素空間 $\mathbb{C}^2$ の 2 重単位開円板 $\mathbb{D}^2$ 上の Hardy 空間を $H^2(\mathbb{D}^2)$ で表す。

定義 2. $H^2(\mathbb{D}^2)$ の閉部分空間 M が不変であるとは、$zM \subset M$ かつ $wM \subset M$ を満たすときにいう。また、集合 $E \subset H^2(\mathbb{D}^2)$ を含む最小の不変部分空間を $[E]$ で表す。

2 変数では座標関数が 2 つあるので、Beurling property は $[M \ominus [zM + wM]]_{\mathcal{H}} = M$ と置き換えられる。すでにこれまでに $H^2(\mathbb{D}^2)$ の不変部分

空間で Beurling property を持たないものが存在することが知られている。しかし、それらはすべてランクが 2 以上の不変部分空間であり、ランク 1 の不変部分空間については明らかになっていない。その中、1991 年に Nakazi によって次の問題が提出された：

問題 3. 任意の関数 $f \in H^2(\mathbb{D}^2)$ によって生成される不変部分空間 $M_f = [f]$ に対して、

$$M_f \ominus [zM_f + wM_f] = \mathbb{C} \cdot g, \quad g \neq 0$$

かつ $M_f = [g]$ を満たすか。

本講演では、この問題について考える。

2. M_f と $H^2(d\mu)$ の関係

非零関数 $f \in H^2(\mathbb{D}^2)$ に対して、

$$d\mu = |f|^2 \, dm \quad \text{on } \mathbb{T}^2$$

とおく。ここで、dm は $\mathbb{T}^2$ 上の正規化されたルベーグ測度である。さらに、$\mathbb{C}^2$ 上の多項式環 $\mathcal{C}$ の $L^2(d\mu)$-閉包を $H^2(d\mu)$ によって表記する。

命題 4. 任意の $f \in H^2(\mathbb{D}^2)$ に対して、$M_f = fH^2(d\mu)$ である。

定義 5. (1) $\lambda \in \mathbb{D}^2$ に対する $H^2(d\mu)$ の再生核 $K_\mu^\lambda \in H^2(d\mu)$ は、

$$f(\lambda) = \langle f, K_\mu^\lambda \rangle_{H^2(d\mu)} \quad \text{for all } f \in H^2(d\mu)$$

を満たす関数である。
(2) 関数 $f \in H^2(d\mu)$ が巡回ベクトルであるとは、$f\mathcal{C}$ が $H^2(d\mu)$ で稠密であるときにいう。

命題 6. $f \in H^2(\mathbb{D}^2)$ とする。そのとき、$[fK_\mu^\lambda] = M_f$ を満たすことと、再生核 K_μ^λ が $H^2(d\mu)$ の巡回ベクトルであることは同値である。

3. 結果

[4] で、具体的な反例を提示することにより、以下が成り立つことが明らかとなった：

定理 7. $H^2(d\mu)$ が巡回ベクトルでない再生核を持つような $f \in H^2(\mathbb{D}^2)$ が存在する。

実際には、上記の結果より強い次の結果が成り立つ例を提示している：

定理 8. $H^2(d\mu)$ が $\mathbb{D}^2$ 上に零集合を持つ再生核を持つような $f \in H^2(\mathbb{D}^2)$ が存在する。

Nakazi の問題は原点における生成核の巡回性と同じであるが、原点における生成核が巡回ベクトルではない例も提示できた。これらの結果と、命題 4，6 を組み合わせることにより、次が成り立つことがわかる：

定理 9. $f \in H^2(\mathbb{D}^2)$ とし、$M_f = [f]$ とする。そのとき、

$$[M_f \ominus [zM_f + wM_f]] \neq M_f$$

を満たす関数 f が存在する。

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正凸錘の部分空間について

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概 要

A. A. Ungar は（可換）群の一般化として (gyrocommutative) gyrogroup を定義し，特殊相対論における速度全体やポアンカレ円板の持つ双曲幾何構造についての研究を行っている（[3]）．また単位的 C^*- 環の正凸錘も (gyrocommutative) gyrogroup としての構造を持ち，その (gyrocommutative) gyrogroup としての構造と双曲幾何構造とがマッチしていることが確認できる（[2]，[1]）．これらの対象は，いずれも群をなしていないにもかかわらず自然に線形空間に類似した構造を持っている．この"線形空間に類似した構造"に注目し，正凸錘の構造について調べてみる．

1　導入

はじめに (gyrocommutative) gyrogroup の定義を行う．

定義 1.1 空でない集合 X とその上の二項演算 $\circ : X \times X \to X$；$(a,b) \mapsto a \circ b$ を併せて考えた $(X, \circ)$ を *magma* という．写像 ϕ: $X \to X$ が全単射で任意の $a, b \in X$ に対して $\phi(a \circ b) = \phi(a) \circ \phi(b)$ を満たすとき $(X, \circ)$ 上の自己同型写像であるという．$(X, \circ)$ 上の自己同型写像全体の集合を $\mathrm{Aut}(X, \circ)$ で表す．

定義 1.2 *Magma* $(X, \oplus)$ が次の条件 (G1) から (G5) を全て満たすとき，$(X, \oplus)$ は *gyrogroup* であるという．さらに条件 (G6) も満たすとき，$(X, \oplus)$ は *gyrocommutative gyrogroup* であるという．

(G1) 単位元 e が存在する．

(G2) 任意の $a \in G$ に対して，逆元 $\ominus a$ が存在する．

(G3) 任意の $a, b, c \in G$ に対して，次の等式を満たす $\mathrm{gyr}[a,b]c \in G$ が一意的に存在する．

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a,b]c.$$

(G4) 任意の $a, b \in G$ に対して，写像 $c \mapsto \mathrm{gyr}[a,b]c$ は $(G, \oplus)$ 上の自己同型写像である．すなわち，$\mathrm{gyr}[a,b] \in \mathrm{Aut}(G, \oplus)$.

(G5) 任意の $a, b \in G$ に対して，$\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a,b]$.

(G6) 任意の $a, b \in G$ に対して，$a \oplus b = \mathrm{gyr}[a,b](b \oplus a)$.

次に (gyrocommutative) gyrogroup に基づいて線形空間を一般化した gyrolinear space を定義する。これは Ungar の定義した gyrovector space の代数構造についての条件のみを抽出したものである．

定義 1.3 *Gyrocommutative gyrogroup* $(X,\oplus)$ と写像 $\otimes:\mathbb{R}\times X\to X$；$(r,\boldsymbol{x})\mapsto r\otimes\boldsymbol{x}$ の組 $(X,\oplus,\otimes)$ が次の条件 (GL1) から (GL2) を全て満たすとき，$(X,\oplus,\otimes)$ は *gyrolinear space* であるという．

(GL1) 任意の $\boldsymbol{x}\in X$ に対して，$1\otimes\boldsymbol{x}=\boldsymbol{x}$.

(GL2) 任意の $r_1,r_2\in\mathbb{R}$ と $\boldsymbol{x}\in X$ に対して，$(r_1+r_2)\otimes\boldsymbol{x}=(r_1\otimes\boldsymbol{x})\oplus(r_2\otimes\boldsymbol{x})$.

(GL3) 任意の $r_1,r_2\in\mathbb{R}$ と $\boldsymbol{x}\in X$ に対して，$(r_1r_2)\otimes\boldsymbol{x}=r_1\otimes(r_2\otimes\boldsymbol{x})$.

(GL4) 任意の $r\in\mathbb{R}$ と $\boldsymbol{x},\boldsymbol{u},\boldsymbol{v}\in X$ に対して，$\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}](r\otimes\boldsymbol{x})=r\otimes\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}]\boldsymbol{x}$.

(GL5) 任意の $r_1,r_2\in\mathbb{R}$ と $\boldsymbol{v}\in X$ に対して，$\mathrm{gyr}[r_1\otimes\boldsymbol{v},r_2\otimes\boldsymbol{v}]$ は X 上の恒等写像.

Gyrolinear space は線形空間の一般化である。次のように，自然に直線・線分・中点などに対応する概念が定義される。

定義 1.4 *Gyrolinear space* $(X,\oplus,\otimes)$ に対して，

$$L[\boldsymbol{a},\boldsymbol{b}](t):=\boldsymbol{a}\oplus t\otimes(\ominus\boldsymbol{a}\oplus\boldsymbol{b})$$

とする。但し，$\boldsymbol{a},\boldsymbol{b}\in X$, $t\in\mathbb{R}$ である。このとき，$L[\boldsymbol{a},\boldsymbol{b}](\mathbb{R})$ を $\boldsymbol{a}$ と $\boldsymbol{b}$ を通る X 上の *gyroline* という。$L[\boldsymbol{a},\boldsymbol{b}]([0,1])$ を *gyrosegment* $\boldsymbol{ab}$ という。$L[\boldsymbol{a},\boldsymbol{b}](\frac{1}{2})$ を $\boldsymbol{a}$ と $\boldsymbol{b}$ の *gyromidpoint* という。

通常の線形空間においては，gyroline は直線，gyrosegment は線分，gyromidpoint は代数的な中点に対応する。

2 正凸錘のジャイロ構造

単位的 C^*- 環の正凸錘に対して，次のように演算を定義する．

定義 2.1 $\mathscr{A}$ を単位的 C^*- 環，$\mathscr{A}_+^{-1}$ をその正値可逆元全体の集合（正凸錘）とする。$\mathscr{A}_+^{-1}$ に対して，$\oplus:\mathscr{A}_+^{-1}\times\mathscr{A}_+^{-1}\mapsto\mathscr{A}_+^{-1}$ と $\otimes:\mathbb{R}\times\mathscr{A}_+^{-1}\mapsto\mathscr{A}_+^{-1}$ を次のように定義する．任意の $\boldsymbol{a},\boldsymbol{b}\in\mathscr{A}_+^{-1}$ と $r\in\mathbb{R}$ に対して，

$$\begin{aligned}\boldsymbol{a}\oplus\boldsymbol{b} &= \boldsymbol{a}^{\frac{1}{2}}\boldsymbol{b}\boldsymbol{a}^{\frac{1}{2}},\\ r\otimes\boldsymbol{a} &= \boldsymbol{a}^{r}.\end{aligned}$$

事実 2.1 $(\mathscr{A}_+^{-1},\oplus,\otimes)$ は *gyrolinear space* である．

次のように $\mathscr{A}_+^{-1}$ の gyrolinear space としての構造と双曲幾何構造の間には深い関係があることがわかる．

事実 2.2 $\mathscr{A}_+^{-1}$ 上の距離 d を

$$d(\boldsymbol{a},\boldsymbol{b})=\|\log(\boldsymbol{a}\ominus\boldsymbol{b})\|$$

によって定めると，これは *Tompson metric* である。但し，$\|\cdot\|$ は単位的 C^*- 環 $\mathscr{A}$ のノルムとする．また，

$$L[\boldsymbol{a},\boldsymbol{b}](t)=\boldsymbol{a}^{\frac{1}{2}}(\boldsymbol{a}^{\frac{1}{2}}\boldsymbol{b}^{-1}\boldsymbol{a}^{\frac{1}{2}})^{-t}\boldsymbol{a}^{\frac{1}{2}}$$

である．したがって，$\boldsymbol{a}$ と $\boldsymbol{b}$ を通る *gyroline* は $\boldsymbol{a}$ と $\boldsymbol{b}$ を通る（*Tompson metric* についての）測地線と一致している．特に *gyromidpoint* は幾何平均と一致している．

3　部分空間

Gyrolinear space は線形空間の類似物として定義されている．これは線形空間と同様の議論を行えることを期待してのものである．しかし，当然ながら全ての議論を線形空間と同じように行うことは出来ない．どのような点では線形空間と同じ用に議論が出き，どのような点で違いが表れるのか，といった問題は興味深い問題である．ここでは”部分空間” に注目して調べてみる．

まずは部分空間を定義することからはじめる．

定義 3.1 $(X,\oplus,\otimes)$ を *gyrolinear space* とする．X の部分集合 S が X の（*gyrolinear space* としての）部分空間であるとは，S が $\oplus$ と $\otimes$ について閉じていることをいう．すなわち，任意の $\boldsymbol{a},\boldsymbol{b}\in S$ と $r\in\mathbb{R}$ に対して，$\boldsymbol{a}\oplus\boldsymbol{b}\in S$，$r\otimes\boldsymbol{a}\in S$ であるとき S は X の部分空間である．

線形空間と同様の議論により以下のことが確認できる．

事実 3.1 S が $(X,\oplus,\otimes)$ の部分空間であるとき，$(S,\oplus,\otimes)$ は *gyrolinear space* である．

事実 3.2 $\{S_\lambda\}_{\lambda\in\Lambda}$ を $(X,\oplus,\otimes)$ の部分空間族とする．このとき，$\bigcap_{\lambda\in\Lambda} S_\lambda$ も $(X,\oplus,\otimes)$ の部分空間である．

定義 3.2 $(X,\oplus,\otimes)$ を *gyrolinear space*，A を X の部分集合とする．このとき，$(X,\oplus,\otimes)$ の部分空間のうち A を部分集合として含むもの全体の共通部分を $S[A]$ で表し，A から生成される X の部分空間であるという．$S[A]$ は A を含む X の部分空間のうち（包含関係で）最小のものである．表記の簡略化のため，A が有限集合のとき，$S[\{\boldsymbol{a}_1,\cdots,\boldsymbol{a}_k\}]$ を単に $S[\boldsymbol{a}_1,\cdots,\boldsymbol{a}_k]$ と書くことにする．

一つの元から生成される部分空間については通常の線形空間と同様の結果が得られる．

事実 3.3 $(X,\oplus,\otimes)$ を *gyrolinear space* とし，$\boldsymbol{a}\in X$ とする．このとき，

(1) $\boldsymbol{a}$ が $(X,\oplus)$ の単位元であるとき，$S[\boldsymbol{a}]=\{\boldsymbol{a}\}$.

(2) $\boldsymbol{a}$ が $(X,\oplus)$ の単位元でないとき，$S[\boldsymbol{a}]=\{r\otimes\boldsymbol{a};r\in\mathbb{R}\}$．$(S[\boldsymbol{a}],\oplus,\otimes)$ は一次元線形空間である．

次に，二つの元から生成される部分空間について考える．$S[\boldsymbol{a},\boldsymbol{b}]$ について考える際，$\boldsymbol{a},\boldsymbol{b}$ のどちらかが単位元である場合や $\boldsymbol{b}\in S[\boldsymbol{a}]$ である場合は上の事実から簡単にわかってしまうので，このような状況でない $\boldsymbol{a},\boldsymbol{b}$ について考えることにする．通常の線形空間の様子と比較して，次の事が疑問としてあがる．これらは通常の線形空間では成立している事柄である．

疑問 3.1 $(X,\oplus,\otimes)$ を *gyrolinear space* とし，$\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}\in X$ とする．このとき $S[\boldsymbol{a},\boldsymbol{b}]$ と $S[\boldsymbol{c},\boldsymbol{d}]$ は同型だろうか．但し，$\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}$ は X の単位元でないものとし，$\boldsymbol{b}\notin S[\boldsymbol{a}],\boldsymbol{d}\notin S[\boldsymbol{c}]$ とする．

疑問 3.2 $(X,\oplus,\otimes)$ を *gyrolinear space* とし，$\boldsymbol{a},\boldsymbol{b}\in X$ とする．このとき $S[\boldsymbol{a},\boldsymbol{b}]=\{\alpha\otimes\boldsymbol{a}\oplus\beta\otimes\boldsymbol{b};\alpha,\beta\in\mathbb{R}\}$ が成り立つか．

疑問 3.2 について，なぜこのようなことが問題になるのか．部分空間は演算について閉じている必要があるので，$S[\boldsymbol{a},\boldsymbol{b}]=\{\alpha\otimes\boldsymbol{a}\oplus\beta\otimes\boldsymbol{b};\alpha,\beta\in\mathbb{R}\}$ が成り立つためには $S[\boldsymbol{a},\boldsymbol{b}]$ が演算について閉じている必要がある．しかし，一般に gyrolinear space では次の等式が成立しないことがある．

$$
\begin{aligned}
\boldsymbol{a} \oplus \boldsymbol{b} &= \boldsymbol{b} \oplus \boldsymbol{a}, \\
(\boldsymbol{a} \oplus \boldsymbol{b}) \oplus \boldsymbol{c} &= \boldsymbol{a} \oplus (\boldsymbol{b} \oplus \boldsymbol{c}), \\
r \otimes (\boldsymbol{a} \oplus \boldsymbol{b}) &= r \otimes \boldsymbol{a} \oplus r \otimes \boldsymbol{b}.
\end{aligned}
$$

このため，

$$
\begin{aligned}
\mu \otimes \boldsymbol{b} \oplus \lambda \otimes \boldsymbol{a} &\in \{\alpha \otimes \boldsymbol{a} \oplus \beta \otimes \boldsymbol{b}; \alpha, \beta \in \mathbb{R}\}, \\
\nu \otimes (\lambda \otimes \boldsymbol{a} \oplus \mu \otimes \boldsymbol{b}) &\in \{\alpha \otimes \boldsymbol{a} \oplus \beta \otimes \boldsymbol{b}; \alpha, \beta \in \mathbb{R}\}
\end{aligned}
$$

といったことさえ明らかではない．実際，反例があることを次に紹介する．

4　実 2×2 行列の正凸錘

単位的 C^*- 環 $\mathscr{A}$ の正凸錘 $\mathscr{A}_+^{-1}$ が自然に gyrolinear space としての構造を持つことは紹介したが，ここでは特に $\mathscr{A}$ として実 2×2 行列全体を考えることにする．このとき，$\mathscr{A}_+^{-1} = M_2(\mathbb{R})_+^{-1}$ は実 2×2 正値可逆行列全体である．すなわち，$X \oplus Y = X^{\frac{1}{2}} Y X^{\frac{1}{2}}$，$r \otimes X = X^r$ によって $(M_2(\mathbb{R})_+^{-1}, \oplus, \otimes)$ は gyrolinear space である．

次のことは簡単に確認できる．($\mathbb{S}_1$，$\mathbb{S}_2$ がそれぞれ演算について閉じていることは明らかである．)

事実 4.1 $\mathbb{S}_1 = \{X \in M_2(\mathbb{R})_+^{-1}; X \text{ は対角行列}\}$, $\mathbb{S}_2 = \{X \in M_2(\mathbb{R})_+^{-1}; \det X = 1\}$ はそれぞれ $(M_2(\mathbb{R})_+^{-1}, \oplus, \otimes)$ の部分空間である．

ここで，$\mathbb{S}_1$ と $\mathbb{S}_2$ はどちらも 2 元生成である．

事実 4.2 $A = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}, C = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}, D = \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix}$ とする．このとき，$A, B, C, D \in M_2(\mathbb{R})_+^{-1}$ であって，

$$
\begin{aligned}
\mathbb{S}_1 &= \left\{ \begin{pmatrix} e^\alpha & 0 \\ 0 & e^\beta \end{pmatrix}; \alpha, \beta \in \mathbb{R} \right\} \\
&= \left\{ \begin{pmatrix} e^\alpha & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & e^\beta \end{pmatrix}; \alpha, \beta \in \mathbb{R} \right\} \\
&= \{\alpha \otimes A \oplus \beta \otimes B; \alpha, \beta \in \mathbb{R}\} \\
\mathbb{S}_2 &= \left\{ \begin{pmatrix} e^\gamma \cosh \delta & \sinh \delta \\ \sinh \delta & e^{-\gamma} \cosh \delta \end{pmatrix}; \gamma, \delta \in \mathbb{R} \right\} \\
&= \left\{ \begin{pmatrix} e^\gamma & 0 \\ 0 & e^{-\gamma} \end{pmatrix} \oplus \begin{pmatrix} \cosh \delta & \sinh \delta \\ \sinh \delta & \cosh \delta \end{pmatrix}; \gamma, \delta \in \mathbb{R} \right\} \\
&= \{\gamma \otimes C \oplus \delta \otimes D; \gamma, \delta \in \mathbb{R}\}
\end{aligned}
$$

よって，$\mathbb{S}_1 = S[A, B]$，$\mathbb{S}_2 = S[C, D]$ である．

ここで，$\mathbb{S}_1$ と $\mathbb{S}_2$ が疑問 3.1 に対する反例を与えていることがわかる．

事実 4.3 対角行列同士は可換であることから $(\mathbb{S}_1, \oplus)$ は群を成していることがわかる．さらに，$(\mathbb{S}_1, \oplus, \otimes)$ は通常の意味での 2 次元線形空間になっている．一方，$(\mathbb{S}_2, \oplus)$ は群を成していない．したがって $\mathbb{S}_1$ と $\mathbb{S}_2$ は同型ではないことがわかる．

事実 4.2 を見ると，$S[A,B]=\{\alpha\otimes A\oplus\beta\otimes B;\alpha,\beta\in\mathbb{R}\}$，$\{\gamma\otimes C\oplus\delta\otimes D;\gamma,\delta\in\mathbb{R}\}$ のように，疑問 3.2 に対して肯定的な様子が伺える．しかしここで表れた行列を用いて疑問 3.2 の反例を発見できる．

事実 4.4 $A=\begin{pmatrix}e & 0\\ 0 & 1\end{pmatrix}, D=\begin{pmatrix}\cosh 1 & \sinh 1\\ \sinh 1 & \cosh 1\end{pmatrix}$ とする．$A,D\in M_2(\mathbb{R})_+^{-1}$ である，$S[A,D]$ は A,D を要素としてもち，演算について閉じているので，

$$\{\alpha\otimes A\oplus\delta\otimes D;\alpha,\delta\in\mathbb{R}\}\cup\{\mu\otimes D\oplus\lambda\otimes A;\lambda,\mu\in\mathbb{R}\}\subset S[A,D]$$

である。しかし，

- $\{\alpha\otimes A\oplus\delta\otimes D;\alpha,\delta\in\mathbb{R}\}\not\subset\{\mu\otimes D\oplus\lambda\otimes A;\lambda,\mu\in\mathbb{R}\}$
- $\{\mu\otimes D\oplus\lambda\otimes A;\lambda,\mu\in\mathbb{R}\}\not\subset\{\alpha\otimes A\oplus\delta\otimes D;\alpha,\delta\in\mathbb{R}\}$

であることがわかるので，$S[A,D]\neq\{\alpha\otimes A\oplus\delta\otimes D;\alpha,\delta\in\mathbb{R}\},\{\mu\otimes D\oplus\lambda\otimes A;\lambda,\mu\in\mathbb{R}\}$ である．

証明 1 $\alpha\otimes A\oplus\delta\otimes D=\mu\otimes D\oplus\lambda\otimes A$ となるための $\alpha,\delta,\lambda,\mu$ の必要条件を与える．そのことから上の事実は明らかである．

$\alpha\otimes A\oplus\delta\otimes D=\mu\otimes D\oplus\lambda\otimes A$ と仮定する．$X,Y\in M_2(\mathbb{R})_+^{-1}$ に対して，$X\oplus Y=X^{\frac{1}{2}}YX^{\frac{1}{2}}$ であったので，$\det(X\oplus Y)=\det X\det Y$ である．ここで $r\otimes A=\begin{pmatrix}e^r & 0\\ 0 & 1\end{pmatrix}$，$s\otimes D=\begin{pmatrix}\cosh s & \sinh s\\ \sinh s & \cosh s\end{pmatrix}$ であることから，$\det(r\otimes A)=e^r$，$\det(s\otimes D)=1$．したがって $\det(\alpha\otimes A\oplus\delta\otimes D)=e^\alpha$，$\det(\mu\otimes D\oplus\lambda\otimes A)=e^\lambda$ である．よって仮定より $\alpha=\lambda$ である．したがって $\alpha\otimes A\oplus\delta\otimes D=\mu\otimes D\oplus\alpha\otimes A$．ここで，

$$\begin{aligned}\alpha\otimes A\oplus\delta\otimes D &= \begin{pmatrix}e^\alpha\cosh\delta & e^{\frac{\alpha}{2}}\sinh\delta\\ e^{\frac{\alpha}{2}}\sinh\delta & \cosh\delta\end{pmatrix}\\ \mu\otimes D\oplus\alpha\otimes A &= \begin{pmatrix}e^\alpha\cosh^2\frac{\mu}{2}+\sinh^2\frac{\mu}{2} & (e^\alpha+1)\cosh\frac{\mu}{2}\sinh\frac{\mu}{2}\\ (e^\alpha+1)\cosh\frac{\mu}{2}\sinh\frac{\mu}{2} & \cosh^2\frac{\mu}{2}+e^\alpha\sinh^2\frac{\mu}{2}\end{pmatrix}\end{aligned}$$

$\alpha\otimes A\oplus\delta\otimes D$ の $(1,1)$ 成分は $(2,2)$ 成分の e^α 倍になっている．したがって，$\mu\otimes D\oplus\alpha\otimes A$ の $(1,1)$ 成分は $(2,2)$ 成分の e^α 倍になる．すなわち，

$$e^\alpha\cosh^2\frac{\mu}{2}+\sinh^2\frac{\mu}{2}=e^\alpha(\cosh^2\frac{\mu}{2}+e^\alpha\sinh^2\frac{\mu}{2}).$$

これを整理すれば，$(e^{2\alpha}-1)\sinh^2\frac{\mu}{2}=0$ より，$\alpha=0$ または $\mu=0$ である．

$\alpha=0$ のとき，$\alpha\otimes A\oplus\delta\otimes D=\mu\otimes D\oplus\alpha\otimes A$ は $\delta\otimes D=\mu\otimes D$ と書き換えられる．したがって $\delta=\mu$ である．

$\mu=0$ のとき，$\mu\otimes D\oplus\alpha\otimes A=\alpha\otimes A=\begin{pmatrix}e^\alpha & 0\\ 0 & 1\end{pmatrix}$ より，$\alpha\otimes A\oplus\delta\otimes D=\mu\otimes D\oplus\alpha\otimes A$ は $\begin{pmatrix}e^\alpha\cosh\delta & e^{\frac{\alpha}{2}}\sinh\delta\\ e^{\frac{\alpha}{2}}\sinh\delta & \cosh\delta\end{pmatrix}=\begin{pmatrix}e^\alpha & 0\\ 0 & 1\end{pmatrix}$ とかける．明らかに $\delta=0$ である．

以上のことから，$\alpha\otimes A\oplus\delta\otimes D=\mu\otimes D\oplus\lambda\otimes A$ となるための $\alpha,\delta,\lambda,\mu$ の必要条件は *(1)*$\alpha=\lambda=0$ かつ $\delta=\mu$，または，*(2)*$\alpha=\lambda$ かつ $\delta=\mu=0$ である．実際にはこれが必要十分条件であることも簡単に確認できる． □

疑問 3.2 の反例があがったことから gyrolunear space について線形空間と同様に，一次結合，一次独立，基底，次元などの議論を行う場合には気をつけなければいけないことがわかる．

最後に，$M_2(\mathbb{R})_+^{-1}$ は 3 元から生成できることを紹介する．

事実 4.5 $A = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}, D = \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix}$ とする．このとき，$A, C, D \in M_2(\mathbb{R})_+^{-1}$ であって，$M_2(\mathbb{R})_+^{-1} = \{\alpha \otimes A \oplus (\gamma \otimes C \oplus \delta \otimes D); \alpha, \gamma, \delta \in \mathbb{R}\}$ である．すなわち，$S[A, C, D] = M_2(\mathbb{R})_+^{-1}$.

証明 2 任意の $X \in M_2(\mathbb{R})_+^{-1}$ が $\alpha \otimes A \oplus (\gamma \otimes C \oplus \delta \otimes D)$ の形で表されることを示せば十分である．$X \in M_2(\mathbb{R})_+^{-1}$ とする．$\alpha = \log \det X$ とおくと，$\det((-\alpha) \otimes A) = e^{-\alpha} = (\det X)^{-1}$ である．ここで，$Y = (-\alpha) \otimes A$ とおけば $Y \in M_2(\mathbb{R})_+^{-1}$ なので，$Y \oplus X \in M_2(\mathbb{R})_+^{-1}$．また $\det(Y \oplus X) = \det Y \det X = 1$ なので $Y \oplus X \in \mathbb{S}_2$ である．したがって事実 *4.2* より，$Y \oplus X = \gamma \otimes C \oplus \delta D$ となる $\gamma, \delta \in \mathbb{R}$ が存在する．また $Y^{-1} \oplus (Y \oplus X) = Y^{-\frac{1}{2}}(Y^{\frac{1}{2}} X Y^{\frac{1}{2}}) Y^{-\frac{1}{2}} = X$ である．$Y^{-1} = \alpha \otimes A$ であるので，$X = \alpha \otimes A \oplus (\gamma \otimes C \oplus \delta \otimes D)$ である． □

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ジャイロベクトル空間やその一般化の公理と部分空間について

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Abstract. メビウスジャイロベクトル空間における有限生成ジャイロベクトル部分空間は, 同じ生成元によって生成される線形部分空間とメビウス球との共通部分と一致することを示す. 応用として, 直交ジャイロ分解の概念を提示し, 直交分解との関係を明らかにする. さらに, 第 2 著者によって最近得られた結果の概略をアナウンスする. 主要な結果のひとつは, メビウスジャイロベクトル空間における直交基底に関する任意元の直交ジャイロ展開である.

1　導入

はじめに抽象的な (gyrocommutative) gyrogroup, gyrovector space の定義を述べる. 詳細や基本的事項は [U] を参照していただきたい.

定義. 空でない集合 G と写像 $\oplus : G \times G \to G$ の組 $(G, \oplus)$ を magma という. $a, b \in G$ に対して $\oplus(a, b)$ を $a \oplus b$ によって表す. $\phi : G \to G$ が magma $(G, \oplus)$ の自己同型であるとは, G から G への全単射で $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$ $(a, b \in G)$ であることをいう. $(G, \oplus)$ の自己同型全体の集合を $\mathrm{Aut}(G, \oplus)$ と表す.

定義 (Gyrocommutative Gyrogroups). [U] magma $(G, \oplus)$ が gyrocommutative gyrogroup であるとは,

(G1)　$\exists 0 \in G \ \text{s.t.}\ 0 \oplus a = a \ \ (\forall a \in G)$

(G2)　$\forall a \in G\, \exists x \in G \ \text{s.t.}\ x \oplus a = 0$

(G3)　$\exists 1 \mathrm{gyr}[a, b]c \in G \ \text{s.t.}\ a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c$

(G4)　$\mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)$

(G5)　$\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$

(G6)　$a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$

を $a, b, c \in G$ に対して満たすことである.

定義 (Gyrovector Spaces). [U] $(G, \oplus, \otimes)$ が real inner product gyrovector space（単に gyrovector space という）であるとは, $(G, \oplus)$ が gyrocommutative gyrogroup で, 実内積空間 $\mathbb{V}$ が存在して $G \subset \mathbb{V}$,

$$\text{(V0)} \quad \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a} \cdot \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b}$$

また, 演算 $\otimes : \mathbb{R} \times G \to G$ が定義されて

(V1) $1 \otimes \boldsymbol{a} = \boldsymbol{a}$

(V2) $(r_1 + r_2) \otimes \boldsymbol{a} = r_1 \otimes \boldsymbol{a} \oplus r_2 \otimes \boldsymbol{a}$

(V3) $(r_1 r_2) \otimes \boldsymbol{a} = r_1 \otimes (r_2 \otimes \boldsymbol{a})$

(V4) $\dfrac{|r| \otimes \boldsymbol{a}}{||r \otimes \boldsymbol{a}||} = \dfrac{\boldsymbol{a}}{||\boldsymbol{a}||}$

(V5) $\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\,(r \otimes \boldsymbol{a}) = r \otimes \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a}$

(V6) $\mathrm{gyr}[r_1 \otimes \boldsymbol{v}, r_2 \otimes \boldsymbol{v}] = I$

(VV) さらに集合 $||G|| = \{\pm ||\boldsymbol{a}||;\, \boldsymbol{a} \in G\} \subset \mathbb{R}$ 上に（別の）演算 $\oplus$, $\otimes$ が定義されて $(||G||, \oplus, \otimes)$ は 1 次元のベクトル空間をなし,

(V7) $||r \otimes \boldsymbol{a}|| = |r| \otimes ||\boldsymbol{a}||$

(V8) $||\boldsymbol{a} \oplus \boldsymbol{b}|| \leq ||\boldsymbol{a}|| \oplus ||\boldsymbol{b}||$

を $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a}, \boldsymbol{b} \in G,\ r_1, r_2, r \in \mathbb{R}$ に対して満たすことである.

例 (Einstein Gyrovector Spaces).[U] c を真空中の光の速さ, 相対論的に許容される質点の速度の全体を $\mathbb{R}^3_c = \{\boldsymbol{a} \in \mathbb{R}^3;\, ||\boldsymbol{a}|| < c\}$ とする. Einstein の速度和とスカラー倍は

$$\boldsymbol{a} \oplus_{\mathrm{E}} \boldsymbol{b} = \frac{1}{1 + \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{c^2}} \left\{ \boldsymbol{a} + \boldsymbol{b} + \frac{1}{c^2} \frac{\gamma_{\boldsymbol{a}}}{1 + \gamma_{\boldsymbol{a}}} \big(\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{b}) \big) \right\}$$

$$r \otimes_{\mathrm{E}} \boldsymbol{a} = c \tanh \left(r \tanh^{-1} \frac{||\boldsymbol{a}||}{c} \right) \frac{\boldsymbol{a}}{||\boldsymbol{a}||} \quad (\text{if } \boldsymbol{a} \neq \boldsymbol{0}), \quad r \otimes_{\mathrm{E}} \boldsymbol{0} = \boldsymbol{0}$$

for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3_c$, $r \in \mathbb{R}$ によって定義される. ここで $\gamma_{\boldsymbol{a}} = \dfrac{1}{\sqrt{1 - \frac{||\boldsymbol{a}||^2}{c^2}}}$.

公理 (VV) の, 集合 $||\mathbb{R}^3_c|| = (-c, c)$ における演算 $\oplus_{\mathrm{E}}, \otimes_{\mathrm{E}}$ は

$$a \oplus_{\mathrm{E}} b = \frac{a + b}{1 + \frac{1}{c^2} ab}$$

$$r \otimes_{\mathrm{E}} a = c \tanh \left(r \tanh^{-1} \frac{a}{c} \right)$$

for all $a, b \in (-c, c)$, $r \in \mathbb{R}$ によって定義される. このとき, $(\mathbb{R}^3_c, \oplus_{\mathrm{E}}, \otimes_{\mathrm{E}})$ は gyrovector space となる. Ungar は任意の実内積空間 $\mathbb{V}$ に対して, 外積の部分を内積で表すことにより, 開球 $\mathbb{V}_c$ へ上記の定義が拡張されることを示している.

例 (Möbius Gyrovector Spaces).[U] $\mathbb{V}$ を任意の実内積空間, 固定された正の数 s に対して $\mathbb{V}_s = \{\boldsymbol{a} \in \mathbb{V}; ||\boldsymbol{a}|| < s\}$ とする. Möbius の和および Möbius のスカラー倍は

$$\boldsymbol{a} \oplus_{\mathrm{M}} \boldsymbol{b} = \frac{\left(1 + \frac{2}{s^2}\boldsymbol{a}\cdot\boldsymbol{b} + \frac{1}{s^2}||\boldsymbol{b}||^2\right)\boldsymbol{a} + \left(1 - \frac{1}{s^2}||\boldsymbol{a}||^2\right)\boldsymbol{b}}{1 + \frac{2}{s^2}\boldsymbol{a}\cdot\boldsymbol{b} + \frac{1}{s^4}||\boldsymbol{a}||^2||\boldsymbol{b}||^2}$$

$$r \otimes_{\mathrm{M}} \boldsymbol{a} = s \tanh\left(r \tanh^{-1}\frac{||\boldsymbol{a}||}{s}\right)\frac{\boldsymbol{a}}{||\boldsymbol{a}||} \quad (\text{if } \boldsymbol{a} \neq \boldsymbol{0}), \qquad r \otimes_{\mathrm{M}} \boldsymbol{0} = \boldsymbol{0}$$

for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}_s$, $r \in \mathbb{R}$ によって定義される. Möbius のスカラー倍と集合 $||\mathbb{V}_s||$ 上の演算は Einstein gyrovector space と同一である.（c が s に替わる.）このとき, $(\mathbb{V}_s, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$ は gyrovector space となる. $\oplus_{\mathrm{M}}, \otimes_{\mathrm{M}}$ をそれぞれ単に $\oplus, \otimes$ と書く.

異なる種類の演算が同一の数式に現れたならば, (1) 通常のスカラー倍 (2) 演算 $\otimes$ (3) 演算 $\oplus$ で優先順を与える, すなわち,

$$r_1 \otimes w_1 \boldsymbol{a}_1 \oplus r_2 \otimes w_2 \boldsymbol{a}_2 = \{r_1 \otimes (w_1 \boldsymbol{a}_1)\} \oplus \{r_2 \otimes (w_2 \boldsymbol{a}_2)\}.$$

そしてこのような場合の括弧は省略する.

一般には, 演算は可換でも, 結合的でも, 分配的でもないことに注意する:

$$\boldsymbol{a} \oplus \boldsymbol{b} \neq \boldsymbol{b} \oplus \boldsymbol{a}$$

$$\boldsymbol{a} \oplus (\boldsymbol{b} \oplus \boldsymbol{c}) \neq (\boldsymbol{a} \oplus \boldsymbol{b}) \oplus \boldsymbol{c}$$

$$r \otimes (\boldsymbol{a} \oplus \boldsymbol{b}) \neq r \otimes \boldsymbol{a} \oplus r \otimes \boldsymbol{b}$$

$$t(\boldsymbol{a} \oplus \boldsymbol{b}) \neq t\boldsymbol{a} \oplus t\boldsymbol{b}.$$

しかし, 左（および右）ジャイロ結合法則 (G3), ジャイロ交換法則 (G6), スカラー分配法則 (V2), スカラー結合法則 (V3) などがあるように, gyrovector space は解明すべき豊かな対称性を有している.

$s \to \infty$ とすると $\mathbb{V}_s$ は全空間 $\mathbb{V}$ に拡大して行き, 演算 $\oplus, \otimes$ は通常のベクトル和, スカラー倍に近づく. これは, 実内積空間における諸結果が Möbius gyrovector spaces における諸結果から復元されうるということを示唆している.

命題. [U]

$$\boldsymbol{a} \oplus \boldsymbol{b} \to \boldsymbol{a} + \boldsymbol{b} \quad (s \to \infty)$$

$$r \otimes \boldsymbol{a} \to r\boldsymbol{a} \quad (s \to \infty).$$

2　有限生成ジャイロベクトル部分空間と直交ジャイロ分解

簡単のため $s=1$ の場合を述べるが, それから任意の $s>0$ に対して対応する結果を導くのは易しい. Möbius gyrovector space では次が成り立つ：

$$\{r_1\otimes\boldsymbol{a}_1 \oplus r_2\otimes\boldsymbol{a}_2\,;\, r_1, r_2 \in \mathbb{R}\} = \{\lambda_1\boldsymbol{a}_1 + \lambda_2\boldsymbol{a}_2;\, \lambda_1, \lambda_2 \in \mathbb{R}\} \cap \mathbb{V}_1$$

for $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{V}_1$.

($\subset$) 演算 $\oplus, \otimes$ の定義から $r_1 \otimes \boldsymbol{a}_1 \oplus r_2 \otimes \boldsymbol{a}_2$ は $\boldsymbol{a}_1, \boldsymbol{a}_2$ の線形結合であり, $\mathbb{V}_1$ が gyrovector space であるということに $\oplus, \otimes$ について閉じていることが含まれているので $r_1 \otimes \boldsymbol{a}_1 \oplus r_2 \otimes \boldsymbol{a}_2 \in \mathbb{V}_1$.
($\supset$) 次の定理による.

定理 1.[AW] Let $(\mathbb{V}_1, \oplus, \otimes)$ be the Möbius gyrovector space and $\boldsymbol{0} \neq \boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{V}_1$. Put $\alpha = \frac{\boldsymbol{a}_1}{||\boldsymbol{a}_1||}\cdot\frac{\boldsymbol{a}_2}{||\boldsymbol{a}_2||}$. Suppose that $0 \neq t_1, t_2 \in \mathbb{R}$ satisfy the condition

$$\left|\left|t_1\frac{\boldsymbol{a}_1}{||\boldsymbol{a}_1||} + t_2\frac{\boldsymbol{a}_2}{||\boldsymbol{a}_2||}\right|\right| < 1.$$

(I) If $2\alpha t_2 + t_1 \neq 0$, then we put

$$c_1 = \frac{{t_1}^2 + 2\alpha t_1 t_2 + {t_2}^2 + 1 - \sqrt{({t_1}^2 + 2\alpha t_1 t_2 + {t_2}^2 + 1)^2 - 8\alpha t_1 t_2 - 4{t_1}^2}}{2(2\alpha t_2 + t_1)}$$

$$c_2 = \frac{{t_1}^2 + 2\alpha t_1 t_2 + {t_2}^2 - 1 + \sqrt{({t_1}^2 + 2\alpha t_1 t_2 + {t_2}^2 + 1)^2 - 8\alpha t_1 t_2 - 4{t_1}^2}}{2t_2}.$$

(II) If $2\alpha t_2 + t_1 = 0$, then we put

$$c_1 = \frac{t_1}{{t_2}^2 + 1}$$

$$c_2 = t_1.$$

Then, we have $0 < |c_1|, |c_2| < 1$ and

$$t_1\frac{\boldsymbol{a}_1}{||\boldsymbol{a}_1||} + t_2\frac{\boldsymbol{a}_2}{||\boldsymbol{a}_2||} = r_1\otimes\boldsymbol{a}_1 \oplus r_2\otimes\boldsymbol{a}_2,$$

where

$$r_1 = \frac{\tanh^{-1} c_1}{\tanh^{-1} ||\boldsymbol{a}_1||} \quad \text{and} \quad r_2 = \frac{\tanh^{-1} c_2}{\tanh^{-1} ||\boldsymbol{a}_2||}.$$

これは次の定理 **2** から導かれる. 定理 **2** の証明で x, y の右辺を導くのは難しくないが, 絶対値を 1 と比較することが重要であり, それなりの議論を要する.

定理 2.[AW] Consider the following system of equations for real numbers:

$$\begin{cases} x^2y^2 + (\gamma x^2 + 2\alpha x - \gamma)y + 1 = 0 & (1) \\ xy^2 + ((2\alpha+\beta)x^2 - \beta)y + x = 0 & (2) \end{cases}$$

Suppose that $-1 \le \alpha \le 1$, $\beta \neq 0$ and $1 + \beta(2\alpha + \beta) < \gamma^2$.

(I) If $2\alpha + \beta \neq 0$, then

$$x = \frac{1 + \beta(2\alpha+\beta) + \gamma^2 - \sqrt{(1 + \beta(2\alpha+\beta) + \gamma^2)^2 - 4(2\alpha+\beta)\beta\gamma^2}}{2(2\alpha+\beta)\gamma}$$
$$y = \frac{1 + \beta(2\alpha+\beta) - \gamma^2 + \sqrt{(1 + \beta(2\alpha+\beta) + \gamma^2)^2 - 4(2\alpha+\beta)\beta\gamma^2}}{2\gamma}$$

is a unique pair as the solution to the system of equations (1), (2), which satisfies $0 < |x|, |y| < 1$. Moreover,

$$x = \frac{1 + \beta(2\alpha+\beta) + \gamma^2 + \sqrt{(1 + \beta(2\alpha+\beta) + \gamma^2)^2 - 4(2\alpha+\beta)\beta\gamma^2}}{2(2\alpha+\beta)\gamma}$$
$$y = \frac{1 + \beta(2\alpha+\beta) - \gamma^2 - \sqrt{(1 + \beta(2\alpha+\beta) + \gamma^2)^2 - 4(2\alpha+\beta)\beta\gamma^2}}{2\gamma}$$

is a unique pair as the solution to the system of equations (1), (2), which satisfies $|x|, |y| > 1$.

(II) If $2\alpha + \beta = 0$, then

$$x = \frac{\beta\gamma}{1 + \gamma^2}$$
$$y = \frac{1}{\gamma}$$

is a unique pair as the solution to the system of equations (1), (2), which satisfies $0 < |x|, |y| < 1$.

定義. $\mathbb{V}_1$ の空でない部分集合 M が gyrovector subspace であるとは, M が演算 $\oplus, \otimes$ について閉じていることをいう, すなわち,

$$\boldsymbol{a}, \boldsymbol{b} \in M,\ \ r \in \mathbb{R} \quad \Rightarrow \quad \boldsymbol{a} \oplus \boldsymbol{b} \in M,\ \ r \otimes \boldsymbol{a} \in M.$$

A を含むような, $\mathbb{V}_1$ のすべての gyrovector subspace の共通部分を A によって生成された gyrovector subspace といい, $\bigvee^g A$ と表す, すなわち,

$$\bigvee{}^g A = \bigcap \{M;\ A \subset M,\ M \text{ is a gyrovector subspace of } \mathbb{V}_1\}.$$

例えば $n=4$, $(i_1, i_2, i_3, i_4) = (1, 4, 2, 3)$ とする. 数式 $\boldsymbol{c}_1 \oplus \boldsymbol{c}_4 \oplus \boldsymbol{c}_2 \oplus \boldsymbol{c}_3$ に, ジャイロ和の順序を特定するため括弧を書き加えるならば, 以下のように 5 つの可能性がある：

$$\boldsymbol{c}_1 \oplus \{\boldsymbol{c}_4 \oplus (\boldsymbol{c}_2 \oplus \boldsymbol{c}_3)\}$$
$$(\boldsymbol{c}_1 \oplus \boldsymbol{c}_4) \oplus (\boldsymbol{c}_2 \oplus \boldsymbol{c}_3)$$
$$\boldsymbol{c}_1 \oplus \{(\boldsymbol{c}_4 \oplus \boldsymbol{c}_2) \oplus \boldsymbol{c}_3\}$$
$$\{\boldsymbol{c}_1 \oplus (\boldsymbol{c}_4 \oplus \boldsymbol{c}_2)\} \oplus \boldsymbol{c}_3$$
$$\{(\boldsymbol{c}_1 \oplus \boldsymbol{c}_4) \oplus \boldsymbol{c}_2\} \oplus \boldsymbol{c}_3$$

定理 3.[AW] Let $(\mathbb{V}_1, \oplus, \otimes)$ be the Möbius gyrovector space, $\mathbf{0} \neq \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n \in \mathbb{V}_1$ and let $(i_1, \cdots, i_n)$ be a permutation of $(1, \cdots, n)$. For an arbitrary given order of gyroaddition for $r_{i_1} \otimes \boldsymbol{a}_{i_1} \oplus \cdots \oplus r_{i_n} \otimes \boldsymbol{a}_{i_n}$, we have the following:

$$\begin{aligned} &\bigvee{}^{g}\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\} \\ &= \{r_{i_1} \otimes \boldsymbol{a}_{i_1} \oplus \cdots \oplus r_{i_n} \otimes \boldsymbol{a}_{i_n};\ r_{i_1}, \cdots, r_{i_n} \in \mathbb{R}\} \\ &= \left\{ t_1 \frac{\boldsymbol{a}_1}{||\boldsymbol{a}_1||} + \cdots + t_n \frac{\boldsymbol{a}_n}{||\boldsymbol{a}_n||};\ t_1, \cdots, t_n \in \mathbb{R} \right\} \cap \mathbb{V}_1. \end{aligned}$$

注意. Einstein gyrovector space の有限生成な gyrovector subspace についても同様である.

次に, 直交ジャイロ分解について述べる. 通常の直交分解から具体的かつ容易に求めることができる. 繰り返しになるが, $s=1$ の場合から一般の $s>0$ での結果を導くことも易しい.

定理 4.[AW] Let $\mathbb{V}$ be a real Hilbert space and let $(\mathbb{V}_1, \oplus, \otimes)$ be the Möbius gyrovector space, and let M be a gyrovector subspace of $\mathbb{V}_1$ that is topologically relatively closed. Suppose that

$$\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_2, \quad \boldsymbol{x}_1 \in \mathrm{clin} M,\ \ \boldsymbol{x}_2 \in M^{\perp}$$

is the (ordinary) orthogonal decomposition of an arbitrary element $\boldsymbol{x} \in \mathbb{V}_1$ with respect to $\mathrm{clin}M$, which is the closed linear subspace generated by M. Then, a unique pair $(\boldsymbol{y}, \boldsymbol{z})$ exists that satisfies

$$\boldsymbol{x} = \boldsymbol{y} \oplus \boldsymbol{z}, \quad \boldsymbol{y} \in M,\ \ \boldsymbol{z} \in M^{\perp} \cap \mathbb{V}_1.$$

Moreover, if $\boldsymbol{x}_1, \boldsymbol{x}_2 \neq \mathbf{0}$, then these elements $\boldsymbol{y}, \boldsymbol{z}$ are determined by

$$\boldsymbol{y} = \lambda_1 \boldsymbol{x}_1,\ \ \boldsymbol{z} = \lambda_2 \boldsymbol{x}_2,$$

where

$$\lambda_1 = \frac{||\boldsymbol{x}_1||^2 + ||\boldsymbol{x}_2||^2 + 1 - \sqrt{(||\boldsymbol{x}_1||^2 + ||\boldsymbol{x}_2||^2 + 1)^2 - 4||\boldsymbol{x}_1||^2}}{2||\boldsymbol{x}_1||^2}$$

$$\lambda_2 = \frac{||\boldsymbol{x}_1||^2 + ||\boldsymbol{x}_2||^2 - 1 + \sqrt{(||\boldsymbol{x}_1||^2 + ||\boldsymbol{x}_2||^2 + 1)^2 - 4||\boldsymbol{x}_1||^2}}{2||\boldsymbol{x}_2||^2}.$$

In addition, the inequalities $0 < \lambda_1 < 1$ and $\lambda_2 > 1$ hold.

注意. 上記の M が Ungar によって導入された Poincaré の距離 h に関して閉ならば, ノルムに関して相対閉であることが分かるので, 定理が適用可能である.

注意. Einstein gyrovector space でも対応する結果が得られる.

3 ジャイロ線形独立性

以下の節では, 第 2 著者によって得られた結果の概要を述べる. gyrovector space と Goebel and Reich による Hilbert 球との関係についても触れる.

定義. 有限集合 $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\} \subset \mathbb{V}_s$ がジャイロ線形独立であるとは, $\{1, \cdots, n\}$ のいかなる置換 $(i_1, \cdots, i_n)$ といかなるジャイロ和の順序に対しても

$$r_{i_1} \otimes \boldsymbol{a}_{i_1} \oplus \cdots \oplus r_{i_n} \otimes \boldsymbol{a}_{i_n} = \boldsymbol{0} \quad \Rightarrow \quad r_1 = \cdots = r_n = 0$$

が成り立つことと定義する.

例. $\mathbb{R}^2$ を複素平面 $\mathbb{C}$ と同一視すると, $\mathbb{R}^2_1$ における演算は $a \oplus b = \dfrac{a+b}{1+\overline{a}b}$ となる.

$$a = \frac{i}{2}, \qquad b = -\frac{2}{5} - \frac{2}{5}i, \qquad c = \frac{1}{2},$$

とすると, このとき

$$a \oplus (b \oplus c) = 0, \qquad (a \oplus b) \oplus c = \frac{4+16i}{53-8i}$$

となり, 組 $\{a, b, c\}$ はジャイロ線形独立ではない. **定理 1** によって

$$r_1 = \frac{\tanh^{-1} \frac{-33+\sqrt{689}}{20}}{\tanh^{-1} \frac{1}{2}} \qquad \text{and} \qquad r_2 = \frac{\tanh^{-1} \frac{17-\sqrt{689}}{20}}{\tanh^{-1} \frac{1}{2}}$$

とおくと,

$$r_1 \otimes c \oplus r_2 \otimes a = \frac{-33+\sqrt{689}}{20} \oplus \frac{17-\sqrt{689}}{20} i = b$$

が分かる.

定理 5.[W2] $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\} \subset \mathbb{V}_s$ を線形独立とする. 2 つのジャイロ線形結合 $r_1 \otimes \boldsymbol{a}_1 \oplus \cdots \oplus r_n \otimes \boldsymbol{a}_n$, $\lambda_1 \otimes \boldsymbol{a}_1 \oplus \cdots \oplus \lambda_n \otimes \boldsymbol{a}_n$ が同じジャイロ和の順序をもち,

$$r_1 \otimes \boldsymbol{a}_1 \oplus \cdots \oplus r_n \otimes \boldsymbol{a}_n = \lambda_1 \otimes \boldsymbol{a}_1 \oplus \cdots \oplus \lambda_n \otimes \boldsymbol{a}_n$$

であるとする. このとき $r_j = \lambda_j \ (j = 1, \cdots, n)$ が成り立つ.

定理 6.[W2] $\mathbb{V}_s$ の有限部分集合に対して, 線形独立とジャイロ線形独立の概念は一致する.

4　Hadamard 空間, 特に Hilbert ball との関係

関数環研究集会での講演のときに, 瀬戸さんから, 高阪さん他が関係している Hilbert ball 等との関係はどうなっているのかという主旨のご質問をいただき, さらに数日後, [B], [GR] をはじめとする文献等の情報をいただきました. この場を借りて感謝致します.

まず, Ungar によって導入された gyrovector space の側のことを述べる.

定義 (Ungar).[U] Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ 上で d と h が

$$d(\boldsymbol{a}, \boldsymbol{b}) = ||\ominus \boldsymbol{a} \oplus \boldsymbol{b}|| = ||\boldsymbol{b} \ominus \boldsymbol{a}||$$

$$h(\boldsymbol{a}, \boldsymbol{b}) = \tanh^{-1} \frac{d(\boldsymbol{a}, \boldsymbol{b})}{s}$$

for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}_s$ によって定義され, $(\mathbb{V}_s, h)$ は距離空間となる. さらに $(\mathbb{V}, || \cdot ||)$ が完備ならば, $(\mathbb{V}_s, h)$ も完備である. ここで $\boldsymbol{b} \ominus \boldsymbol{a}$ は $\boldsymbol{b} \oplus (\ominus \boldsymbol{a})$ を表す.

$\oplus$ を $\oplus_{\mathrm{E}}$ に取り替えることによって, Einstein gyrovector space $(\mathbb{V}_s, \oplus_{\mathrm{E}}, \otimes_{\mathrm{E}})$ 上で d_{E} と h_{E} が定義され, 同様のことが成り立つ.

補題.

$$||\boldsymbol{a} \oplus \boldsymbol{b}||^2 = \frac{||\boldsymbol{a}||^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} + ||\boldsymbol{b}||^2}{1 + \frac{2}{s^2}\boldsymbol{a}\cdot\boldsymbol{b} + \frac{1}{s^4}||\boldsymbol{a}||^2||\boldsymbol{b}||^2}$$

$$||\boldsymbol{a} \oplus_{\mathrm{E}} \boldsymbol{b}||^2 = \frac{1}{\left(1 + \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{s^2}\right)^2} \left\{ ||\boldsymbol{a}||^2 + ||\boldsymbol{b}||^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} - \frac{1}{s^2}||\boldsymbol{a}||^2||\boldsymbol{b}||^2 + \frac{1}{s^2}(\boldsymbol{a}\cdot\boldsymbol{b})^2 \right\}$$

for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}_s$.

Goebel と Reich による Hilbert ball の定義は次の通りである.

定義 (Goebel and Reich).[GR] $\mathcal{H}$ を複素 Hilbert 空間とし $\mathbb{B} = \{x \in \mathcal{H}; ||x|| < 1\}$ と書く.

$$\rho(x, y) = \tanh^{-1} (1 - \sigma(x, y))^{\frac{1}{2}},$$

where

$$\sigma(x, y) = \frac{(1 - ||x||^2)(1 - ||y||^2)}{|1 - \langle x, y \rangle|^2}.$$

for any $x, y \in \mathbb{B}$.

定理 (Goebel and Reich).[GR] $(\mathbb{B}, \rho)$ は Hadamard 空間 (完備 CAT(0) 空間) である.

$\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}_s$ に対して, 記号 ρ および σ を

$$\rho(\boldsymbol{a}, \boldsymbol{b}) = \tanh^{-1} (1 - \sigma(\boldsymbol{a}, \boldsymbol{b}))^{\frac{1}{2}},$$

および

$$\sigma(\boldsymbol{a},\boldsymbol{b}) = \frac{(s^2-||\boldsymbol{a}||^2)\,(s^2-||\boldsymbol{b}||^2)}{(s^2-\boldsymbol{a}{\cdot}\boldsymbol{b})^2}$$

と流用する.

定理. $\boldsymbol{a},\boldsymbol{b}\in\mathbb{V}_s$ に対して次の等式が成り立つ:

(i) $\dfrac{||\boldsymbol{a}\ominus_{\mathrm{E}}\boldsymbol{b}||}{s} = (1-\sigma(\boldsymbol{a},\boldsymbol{b}))^{\frac{1}{2}}$

(ii) $h_{\mathrm{E}}(\boldsymbol{a},\boldsymbol{b}) = \tanh^{-1}\dfrac{||\boldsymbol{a}\ominus_{\mathrm{E}}\boldsymbol{b}||}{s} = \rho(\boldsymbol{a},\boldsymbol{b})$

(iii) $2h(\boldsymbol{a},\boldsymbol{b}) = 2\tanh^{-1}\dfrac{||\boldsymbol{a}\ominus\boldsymbol{b}||}{s} = \rho(2\otimes\boldsymbol{a},2\otimes\boldsymbol{b})$.

$\mathbb{V}$ が実 Hilbert 空間ならば, Ungar の Einstein gyrovector space $(\mathbb{V}_1,h_{\mathrm{E}})$ と Goebel and Reich の Hilbert ball $(\mathbb{B},\rho)$ は距離空間として一致する. そして, 2 元のジャイロ凸結合は測地距離空間 $(\mathbb{B},\rho)$ における凸結合に他ならない.

定義. gyrovector space $(G,\oplus,\otimes)$ の空でない部分集合 A が gyroconvex であるとは, 次が成り立つことをいう.

$$\boldsymbol{a},\boldsymbol{b}\in A,\ \ 0\le r\le 1 \quad\Rightarrow\quad \boldsymbol{a}\oplus r\otimes(\ominus\boldsymbol{a}\oplus\boldsymbol{b})\in A.$$

Möbius gyrovector space における Poincaré の距離 h に関して閉である gyroconvex subset の最良近似性を示す次の定理は, 講演で述べたように, Ungar によるジャイロ中線定理を使って証明できるが, Hilbert ball に関する結果と Möbius および Einstein gyrovector space が同型であることからも分かる.

定理 (Goebel and Reich).[GR] $\mathbb{V}$ を実 Hilbert 空間, A を Möbius gyrovector space $(\mathbb{V}_s,\oplus,\otimes)$ の距離 h に関する閉ジャイロ凸部分集合とする. このとき, 任意の元 $\boldsymbol{x}\in\mathbb{V}_s$ に対し, $\boldsymbol{y}\in A$ が

$$h(\boldsymbol{x},\boldsymbol{y}) = \inf_{\boldsymbol{a}\in A} h(\boldsymbol{x},\boldsymbol{a})$$

を満たすように一意に存在する.

5　メビウスジャイロベクトル空間における直交ジャイロ展開

この節では, ジャイロベクトル部分空間に対して, それをジャイロ凸集合と考えて既知の定理を適用しても得られないような結果について述べる. 以下はすべて, メビウスジャイロベクトル空間におけるものである.

補題. $A^{\perp} \cap \mathbb{V}_s$ は h-閉ジャイロベクトル部分空間.

補題. $\boldsymbol{a}_n$, $\boldsymbol{a} \in \mathbb{V}_s$ かつ $||\boldsymbol{a}_n - \boldsymbol{a}|| \to 0$ ならば, $h(\boldsymbol{a}_n, \boldsymbol{a}) \to 0$.

補題. M がジャイロベクトル部分空間ならば, 距離 h に関する閉包 $\overline{M}^h$ もまたジャイロベクトル部分空間.

補題. 有限生成ジャイロベクトル部分空間は距離 h に関して閉.

補題. $\mathbb{V}$ を実 Hilbert 空間とする. A が $(\mathbb{V}_s, h)$ で閉ならば, A は $(\mathbb{V}_s, ||\cdot||)$ で相対閉. したがって, M が $\mathbb{V}_s$ の h-閉ジャイロベクトル部分空間ならば, 直交ジャイロ分解の定理 **4** が適用可能である.

定理 7.[W2] $\mathbb{V}$ を実 Hilbert 空間, M を $\mathbb{V}_s$ の h-閉ジャイロベクトル部分空間, $\boldsymbol{x} \in \mathbb{V}_s$ とする.

(1) $\boldsymbol{x} = \boldsymbol{y} \oplus \boldsymbol{z}$, $\boldsymbol{y} \in M$, $\boldsymbol{z} \in M^{\perp} \cap \mathbb{V}_s$ を M に関する直交ジャイロ分解とする. このとき, $\boldsymbol{y}$ は M の元として h に関する $\boldsymbol{x}$ の最近点である. すなわち $\boldsymbol{y}$ は次の等式を満たす：

$$h(\boldsymbol{x}, \boldsymbol{y}) = \inf_{\boldsymbol{m} \in M} h(\boldsymbol{x}, \boldsymbol{m}). \tag{3}$$

(2) 逆に, $\boldsymbol{y}$ が M の元として h に関する $\boldsymbol{x}$ の最近点であるとする, すなわち $\boldsymbol{y} \in M$ で等式 (3) を満たすとする. このとき,

$$\boldsymbol{x} = \boldsymbol{y} \oplus (\ominus \boldsymbol{y} \oplus \boldsymbol{x})$$

は M に関する直交ジャイロ分解である. すなわち $\ominus \boldsymbol{y} \oplus \boldsymbol{x} \in M^{\perp} \cap \mathbb{V}_s$.

定義. (i) $\{\boldsymbol{a}_n\}_n$ を $\mathbb{V}_s$ 内の列とする. 級数

$$\Big(\big((\boldsymbol{a}_1 \oplus \boldsymbol{a}_2) \oplus \boldsymbol{a}_3 \big) \oplus \cdots \oplus \boldsymbol{a}_n \Big) \oplus \cdots$$

が収束するとは, $\boldsymbol{x} \in \mathbb{V}_s$ が存在して $h(\boldsymbol{x}, \boldsymbol{x}_n) \to 0 \ (n \to \infty)$ を満たすことをいう. ここで列 $\{\boldsymbol{x}_n\}_n$ は $\boldsymbol{x}_1 = \boldsymbol{a}_1$ および $\boldsymbol{x}_n = \boldsymbol{x}_{n-1} \oplus \boldsymbol{a}_n$ によって帰納的に定められるものとする.

(ii) $\{a_n\}_n$ を $|a_n| < s$ なる実数列とする. 級数

$$\sum_{n=1}^{\infty}{}^{\oplus} a_n = a_1 \oplus a_2 \oplus \cdots \oplus a_n \oplus \cdots$$

が収束するとは, $|x| < s$ なる実数 x が存在して $x_n \to x$ であることをいう. ここで列 $\{x_n\}_n$ は $x_1 = a_1$ および $x_n = x_{n-1} \oplus a_n$ によって帰納的に定められるものとする.

次の補題は [U, (3.147), (3.148)] から直ちに分かることであるが, キーポイントのひとつである.

補題. $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} \subset \mathbb{V}_s$ が直交系ならば $\oplus$ は結合的である, すなわち

$$\boldsymbol{u} \oplus (\boldsymbol{v} \oplus \boldsymbol{w}) = (\boldsymbol{u} \oplus \boldsymbol{v}) \oplus \boldsymbol{w}.$$

補題によって次の定理の (i) では括弧が不要.

定理 8.[W2] $\{\boldsymbol{e}_n\}_{n=1}^{\infty}$ を実 Hilbert 空間 $\mathbb{V}$ の正規直交列とする. $\{w_n\}_{n=1}^{\infty}$ を $0 < w_n < s$ なる実数列とする. 任意の実数列 $\{r_n\}_{n=1}^{\infty}$ に対して以下は同値:

(i) 級数

$$r_1 \otimes w_1 \boldsymbol{e}_1 \oplus r_2 \otimes w_2 \boldsymbol{e}_2 \oplus \cdots \oplus r_n \otimes w_n \boldsymbol{e}_n \oplus \cdots$$

はある元 $\boldsymbol{x} \in \mathbb{V}_s$ に収束する.

(ii) 級数 $\displaystyle\sum_{n=1}^{\infty}{}^{\oplus} \frac{(r_n \otimes w_n)^2}{s}$ は $|x| < s$ なる実数 x に収束する.

例. $s = 1$ とする. 実数列 $a_n = \dfrac{1}{2n}$ を考える.

$$x_n = a_1 \oplus \cdots \oplus a_n = 1 - \frac{1}{n+1} \qquad (n = 1, 2, \cdots).$$

$r_n = \dfrac{\tanh^{-1} \frac{1}{\sqrt{2n}}}{\tanh^{-1} \frac{1}{2}}$ とおく. このとき, $r_n \otimes \dfrac{1}{2} = \tanh\left(r_n \tanh^{-1} \dfrac{1}{2}\right) = \dfrac{1}{\sqrt{2n}}$ となり,

$$\sum_{j=1}^{n}{}^{\oplus} \left(r_j \otimes \frac{1}{2}\right)^2 = \sum_{j=1}^{n}{}^{\oplus} \frac{1}{2j} = 1 - \frac{1}{n+1},$$

これは $|x| < 1$ であるような実数 x には収束しない. この例は $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n} = \infty$ に対応していると考えられる.

問題. $\displaystyle\sum_{n=1}^{\infty}{}^{\oplus} \frac{1}{(2n)^2} = \ ?$

$a, b > 0$ ならば $\quad a \oplus b = \dfrac{a+b}{1+ab} < a + b \quad$ であることから,

$$\sum_{n=1}^{\infty}{}^{\oplus} \frac{1}{(2n)^2} < \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \cdot \frac{\pi^2}{6} \approx 0.411234$$

なので, この文脈の（$\oplus$ の）級数として収束していることは確かである.

定理 9.[W2] $\{\boldsymbol{e}_n\}_{n=1}^{\infty}$ を実 Hilbert 空間 $\mathbb{V}$ の正規直交基底とする. $\{w_n\}_{n=1}^{\infty}$ を $0 < w_n < s$ なる実数列とする. このとき, 任意の $\boldsymbol{x} \in \mathbb{V}_s$ は次のように直交ジャイロ展開される：

$$\boldsymbol{x} = r_1 \otimes w_1 \boldsymbol{e}_1 \oplus r_2 \otimes w_2 \boldsymbol{e}_2 \oplus \cdots \oplus r_n \otimes w_n \boldsymbol{e}_n \oplus \cdots .$$

収束は距離 h に関するものであり, 有限部分和は直交性から演算の順序によらず, 括弧を必要としない. また直交ジャイロ展開の係数 $\{r_n\}_{n=1}^{\infty}$ は具体的な手続きで計算できる.

定理 10.[W2] $\{\boldsymbol{e}_n\}_{n=1}^{\infty}$ を実 Hilbert 空間 $\mathbb{V}$ の正規直交列とする. $\{w_n\}_{n=1}^{\infty}$ を $0 < w_n < s$ なる実数列とする. 以下は互いに同値:

(i) $\{\boldsymbol{e}_n\}_{n=1}^{\infty}$ は（直交系として）完全である

(ii) $\{w_n\boldsymbol{e}_n\}_{n=1}^{\infty}$ が生成する h-閉ジャイロベクトル部分空間は $\mathbb{V}_s$ と一致する

(iii) $||\boldsymbol{x}||^2 = \sum_{n=1}^{\infty}{}^{\oplus} \frac{(r_n \otimes w_n)^2}{s} \quad (\boldsymbol{x} \in \mathbb{V}_s)$

ここで $\{r_n\}_{n=1}^{\infty}$ は定理 **9** の手続きによって決まる実数列である.

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The Toeplitzness of weighted composition operators

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1 Introduction

Let H^2 be the Hardy-Hilbert space of all analytic functions on the open unit disk $\mathbb{D}$ with square-summable Taylor coefficients. For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in H^2, identifying functions in H^2 with their boundary functions, the standard inner product is defined as

$$\begin{aligned}\langle f, g\rangle &= \sum_{n=0}^{\infty} a_n \overline{b_n} \\ &= \int_{\partial\mathbb{D}} f(e^{i\theta})\overline{g(e^{i\theta})}dm(\theta),\end{aligned}$$

where m is the normalized Lebesgue measure on the boundary $\partial\mathbb{D}$ of $\mathbb{D}$. Refer to [6, 10] for the basic properties of the classical Hardy spaces.

Let T be a bounded linear operators on H^2. Then T is a Toeplitz operator if and only if $S^*TS = T$, where S is the forward shift defined by $Sf(z) = zf(z)$ for $z \in \mathbb{D}$ and $f \in H^2$ and S^* is the backward shift on H^2. In the natural way, for a bounded measurable function $u \in L^\infty(\partial\mathbb{D})$, a Toeplitz operator T_u on H^2 is defined as $T_u f = P(uf)$ for $f \in H^2$, where P is the orthogonal projection from $L^2(\partial\mathbb{D})$ to H^2. Recall that the only compact Toeplitz operator on H^2 is the zero operator. See [4, 8] for operator theory on H^2.

In [1], Barría and Halmos called an operator T on H^2 asymptotically Toeplitz if the sequence of operators $\{S^{*^n}TS^n\}$ converges strongly on H^2. Then Feintuch [7] suggested the analogous conditions relative to either weak or norm operator convergence. So there are actually three different kinds of asymptotic toeplitzness.

Definition. Let T be a bounded linear operator on H^2.

(i) T is said to be *uniformly asymptotically Toeplitz* if there is a bounded linear operator A on H^2 such that $\|S^{*^n}TS^n - A\| \to 0$ as $n \to \infty$.

(ii) T is said to be *strongly asymptotically Toeplitz* if there is an operator A on H^2 such that $\|(S^{*^n}TS^n - A)f\| \to 0$ as $n \to \infty$ for any $f \in H^2$.

(iii) T is said to be *weakly asymptotically Toeplitz* if there is an operator A on H^2 such that $\langle(S^{*^n}TS^n - A)f, g\rangle \to 0$ as $n \to \infty$ for all $f, g \in H^2$.

Feintuch [7] showed the following result.

Theorem of Feintuch. A bounded linear operator on H^2 is uniformly asymptotically Toeplitz if and only if it is the sum of a Toeplitz operator and a compact operator.

The asymptotic toeplitzness of composition operators originally was considered by Shapiro. For an analytic self-map φ of $\mathbb{D}$, the composition opertaor C_φ is defined by $C_\varphi f = f \circ \varphi$. It has been known for a long time that such operators are bounded linear operators on H^2. See [2, 11, 14] for the study of composition operators. Nazarov and Shapiro [9] investigated properties of the asymptotic toeplitzness of composition operators and adjoints. Also, refer to [12] and to [13] for a survey of early results on the toeplitzness of composition operaors. Recently the toeplitzness of products of composition operators and their adjoints is independently investigated in [3, 5].

The concept of composition operators has been generalized to weighted composition operators. Let u be a non-zero bounded analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. We difine the weighted composition operator $M_u C_\varphi$ by

$$M_u C_\varphi f = u \cdot f \circ \varphi$$

for $f \in H^2$. Then $M_u C_\varphi$ is a bounded linear operator on H^2.

In this article we here consider the asymptotic toeplitzness associated with weighted composition operators.

2 Toeplitzness of weighted composition operators

First we consider the condition for the weighted composition operator to be a Toeplitz operator.

Theorem 2.1 *Let u be a non-zero bounded analytic function on $\mathbb{D}$ and φ a non-constant analytic self-map of $\mathbb{D}$. Then $M_u C_\varphi$ is Toeplitz if and only if φ is the identity.*

Due to Feintuch's theorem, we can show the following.

Theorem 2.2 *Let u be a non-zero bounded analytic function on $\mathbb{D}$ and φ a non-constant analytic self-map of $\mathbb{D}$. Then $M_u C_\varphi$ is uniformly asymptotically Toeplitz if and only if $M_u C_\varphi$ is compact or φ is the identity.*

The compactness of $M_u C_\varphi$ on H^2 is an interesting problem but is difficult. Now it remains open. The following may be implied.

For a non-constant analytic self-map φ of $\mathbb{D}$, denote $\Gamma(\varphi) = \{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}$, where we are identifying φ with its boundary function.

Theorem 2.3 *Let u be a non-zero bounded analytic function on $\mathbb{D}$ and φ a non-constant analytic self-map of $\mathbb{D}$. If $M_u C_\varphi$ is compact on H^2, then $|\varphi| < 1$ a.e. on $\partial\mathbb{D}$.*

Moreover, assume that u is continuous on a neighborhood $\mathcal{N}$ of $\Gamma(\varphi)$ in $\partial\mathbb{D}$. If $u = 0$ on $\Gamma(\varphi)$, then $M_u C_\varphi$ is compact on H^2.

By the analyticity of u, $m(\Gamma(\varphi)) = 0$, For example, $u(z) = 1 - z$ and $\varphi(z) = (1 + z)/2$ satisfy this condition.

Next we consider the strongly asymptotically Toeplitzness. If $M_u C_\varphi$ is compact, then $M_u C_\varphi$ is uniformly asymptotically Toeplitz and so strongly (weakly) asymptotically Toeplitz.

Theorem 2.4 *Let u be a non-zero bounded analytic function on $\mathbb{D}$ and φ a non-constant analytic self-map of $\mathbb{D}$ such that $M_u C_\varphi$ is not compact. If $|\varphi| < 1$ a.e. on $\partial\mathbb{D}$, then $M_u C_\varphi$ is strongly (and so weakly) asymptotically Toeplitz with asymptotic symbol zero.*

We could obtain the converse of the theorem above under the hypothesis.

Theorem 2.5 *Let u be a non-zero bounded analytic function on $\mathbb{D}$ and φ a non-constant analytic self-map of $\mathbb{D}$ with $\varphi(z) \not\equiv z$. Suppose that $\varphi(0) = 0$. If $M_u C_\varphi$ is strongly asymptotically Toeplitz with asymptotic symbol zero, then $|\varphi| < 1$ a.e. on $\partial\mathbb{D}$.*

Finally we obtain the criterion for $M_u C_\varphi$ to be weakly asymptotically Toeplitz.

Theorem 2.6 *Let u be a non-zero bounded analytic function on $\mathbb{D}$ and φ a non-constant analytic self-map of $\mathbb{D}$. If $M_u C_\varphi$ is weakly asymptotically Toeplitz with asymptotic symbol zero, then φ is not a nontrivial rotation. Furtheremore, if φ is not a rotation with $\varphi(0) = 0$, $M_u C_\varphi$ is weakly asymptotically Toeplitz with asymptotic symbol zero.*

The proof is done by the same way as in [9]. In this case the behavior of the weight u does not cause the weakly asymptotically Toeplitzness.

3 Adjoint asymptotic toeplitzness

In this section we consider the adjoint of $M_u C_\varphi$. But it is easily checked that the Toeplitzness, uniformly asymptotic Toeplitzness and weakly asymptotic Toeplitzness of $(M_u C_\varphi)^*$ are ones of $M_u C_\varphi$.

We could show the following by the same method as in [9].

Theorem 3.1 *Let u be a non-zero bounded analytic function on $\mathbb{D}$ and φ a non-constant analytic self-map of $\mathbb{D}$. Suppose that $\varphi(0) = 0$ and φ is not a rotation. Then $(M_u C_\varphi)^*$ is strongly asymptotically Toeplitz.*

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