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2018 年度の関数環研究集会は，飯田安保氏のご協力により金沢医科大学を会場として，2018 年 11 月 30 日（金）から 12 月 2 日（日）までの期間に開催されました．

今年度の関数環研究集会にも韓国から参加者があり，韓国の大学院生 2 名が講演をしてくださいました．また，新潟大学の学部学生 1 名と大学院生 1 名も講演をしてくださいました．今後も若手の研究者の講演者・参加者が増えていく事を期待しています．

講演者の方々から報告集原稿をお送りいただきましたので，それらを取りまとめて 2018 年度関数環研究集会報告集とさせていただきます．

会場責任者　飯田　安保 (金沢医科大学)
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Conference on Function Algebras 2018

Integral operators acting from Bergman spaces to BMOA-type spaces

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1 Introduction

Throughout let $\mathbb{D}$ be the open unit disk in the complex plane and $\mathcal{H}(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. In the theory of analytic function spaces, Hardy, Bergman and Bloch spaces have been actively investigated as classical examples. And then Zhao introduced the general family of the spaces $F(p, q, s)$ unifying most of the analytic function spaces mentioned above in his thesis [5].

For $a \in \mathbb{D}$, let φ_a be the automorphism of $\mathbb{D}$, defined by

$$\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}$$

and let the Green's function g of $\mathbb{D}$ be

$$g(z, a) = \log \frac{1}{|\varphi_a(z)|}.$$

The pseudo-hyperbolic distance $\rho(z, w)$ between z and w in $\mathbb{D}$ is denoted by

$$\rho(z, w) = |\varphi_z(w)| = \left| \frac{z - w}{1 - \overline{z}w} \right|.$$

Let $0 < p < \infty, -2 < q < \infty$ and $0 < s < \infty$. The space $F(p, q, s)$ is consisting of all $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q g^s(z, a) \, dA(z) < \infty,$$

where $dA(z) = dxdy/\pi$ denotes the Lebesgue area measure on $\mathbb{D}$. The space $F_o(p, q, s)$ is also defined as the set of all $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$\lim_{|a| \to 1} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q g^s(z, a) \, dA(z) = 0.$$

In [5] and [6], Zhao showed that if $s > 1$, $\alpha > 0$, and $q = p\alpha - 2$, then the spaces $F(p, p\alpha - 2, s)$ and $F_o(p, p\alpha - 2, s)$ can be regarded as the Bloch-type space and the little Bloch-type space, respectively.

Let $s = 1$, $\alpha > 0$, and $q = p\alpha - 2$. The space $F(p, p\alpha - 2, 1)$ is called the BMOA-type space (see [5] and [7]). Explicitly, we denote the spaces considered in this manuscript as follows: For $f \in \mathcal{H}(\mathbb{D})$ and $a \in \mathbb{D}$, we put

$$M_p^\alpha(f, a) = \int_{\mathbb{D}} (1 - \rho(a, z)^2)(1 - |z|^2)^{p\alpha - 2} |f'(z)|^p \, dA(z). \tag{1}$$

Let BMOA_p^α be the set of all $f \in \mathcal{H}(\mathbb{D})$ for which

$$\sup_{a \in \mathbb{D}} M_p^\alpha(f, a) < \infty.$$

Then BMOA_p^α is a Banach space under the norm

$$\|f\|_{\mathrm{BMOA}_p^\alpha} = |f(0)| + \left\{ \sup_{a \in \mathbb{D}} M_p^\alpha(f, a) \right\}^{1/p}.$$

Let VMOA_p^α denote the closed subspace of BMOA_p^α consisting of functions f with

$$\lim_{|a| \to 1} M_p^\alpha(f, a) = 0. \tag{2}$$

By [5, Theorems 1.3 and 1.4] or [6, Theorems 1 and 2], BMOA_p^α (respectively, VMOA_p^α) is contained in the Bloch-type space (respectively, the little Bloch-type space). It is known that BMOA_2^1 (respectively, VMOA_2^1) is the classical space BMOA (respectively, VMOA) of analytic functions of bounded (respectively, vanishing) mean oscillation.

For the case $s = 0$, the space $F(p, q, 0)$ is consisting of all $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q \, dA(z) < \infty.$$

The space $F(p, q, 0)$ would be regarded as a weighted Bergman space. For $0 < p < \infty$ and $-1 < \alpha < \infty$, let A_α^p denote the weighted Bergman space of all functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{A_\alpha^p}^p = (1 + \alpha) \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha \, dA(z) < \infty.$$

We remark that $f \in A_\alpha^p$ if and only if

$$\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p + \alpha} \, dA(z) < \infty.$$

(see [8, Theorem 4.28]) The Hardy spaces H^p can be viewed as limiting spaces of weighted Bergman spaces A_α^p as α decreases to -1. Let H^∞ be the Banach algebra of bounded analytic functions f on $\mathbb{D}$ with the norm $\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}$.

For a fixed function $\varphi \in \mathcal{H}(\mathbb{D})$, we define two types of integral operators on $\mathcal{H}(\mathbb{D})$:

$$S_\varphi f(z) = \int_0^z \varphi(\zeta) f'(\zeta) \, d\zeta$$

and

$$T_\varphi f(z) = \int_0^z \varphi'(\zeta) f(\zeta)\, d\zeta.$$

The latter one has attracted interest as a generalized Cesàro or Volterra operator. Moreover, by the equality

$$\varphi(z)f(z) = \varphi(0)f(0) + S_\varphi f(z) + T_\varphi f(z)$$

these operators are related to the multiplication operators.

Now we let $1 \le p < \infty, -1 < \alpha < \infty$ and $0 < \beta < \infty$. We will consider integral operators S_φ and T_φ acting from the weighted Bergman space A^p_α to the BMOA-type space $BMOA^\beta_q$ and the VMOA-type space $VMOA^\beta_q$.

2 Into the space BMOA^β_p

At first we consider the boundedness of S_φ from A^p_α to $BMOA^\beta_q$.

Theorem 2.1 *Let $1 \le p < \infty, -1 < \alpha < \infty$ and $0 < \beta < \infty$.*

(i) $S_\varphi : A^p_\alpha \to \mathrm{BMOA}^\beta_p$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{\beta-1-\frac{2+\alpha}{p}} |\varphi(z)| < \infty.$$

Moreover, this equivalence also holds for any Hardy space H^p with $1 \le p < \infty$.

(ii) $T_\varphi : A^p_\alpha \to \mathrm{BMOA}^\beta_p$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{\beta-\frac{2+\alpha}{p}} |\varphi'(z)| < \infty.$$

Moreover, this equivalence also holds for any Hardy space H^p with $1 \le p < \infty$.

If $p = 2$ and $\beta = 1$, then we get the results for BMOA.

Corollary 2.2 *For $\alpha > -1$, the following hold.*

(i) $S_\varphi : A^2_\alpha \to \mathrm{BMOA}$ is bounded if and only if $\varphi \equiv 0$.

(ii) $T_\varphi : A^2_\alpha \to \mathrm{BMOA}$ is bounded if and only if $\sup_{z\in\mathbb{D}} (1-|z|^2)^{-\frac{\alpha}{2}}|\varphi'(z)| < \infty$ for $-1 < \alpha \le 0$ and φ is constant for $0 < \alpha$.

Moreover, we have the results for the Hardy space H^2, too.

Corollary 2.3 *(i) $S_\varphi : H^2 \to \mathrm{BMOA}$ is bounded if and only if $\varphi \equiv 0$.*

(ii) $T_\varphi : H^2 \to \mathrm{BMOA}$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{\frac{1}{2}} |\varphi'(z)| < \infty.$$

Next, we consider the compacctness of S_φ from A^p_α to $BMOA^\beta_q$.

Theorem 2.4 *Let $1 \le p < \infty, -1 < \alpha < \infty$ and $0 < \beta < \infty$.*

(i) Suppose that $S_\varphi : A^p_\alpha \to \mathrm{BMOA}^\beta_p$ is bounded. Then $S_\varphi : A^p_\alpha \to \mathrm{BMOA}^\beta_p$ is compact if and only if

$$\lim_{|z|\to 1}(1-|z|^2)^{\beta-1-\frac{2+\alpha}{p}}|\varphi(z)| = 0. \tag{3}$$

Moreover, this equivalence also holds for any Hardy space H^p with $1 \le p < \infty$.

(ii) Suppose that $T_\varphi : A^p_\alpha \to \mathrm{BMOA}^\beta_p$ is bounded. Then $T_\varphi : A^p_\alpha \to \mathrm{BMOA}^\beta_p$ is compact if and only if

$$\lim_{|z|\to 1}(1-|z|^2)^{\beta-\frac{2+\alpha}{p}}|\varphi'(z)| = 0.$$

Moreover, this equivalence also holds for any Hardy space H^p with $1 \le p < \infty$.

3 Into the space VMOA^β_p

In this section we will consider the boundedness and compactness of S_φ and T_φ acting to VMOA^β_p. In the sequel we could obtain the following equivalence.

Theorem 3.1 *Let $1 \le p < \infty, -1 < \alpha < \infty$ and $0 < \beta < \infty$. The following are equivalent.*

(i) $S_\varphi : A^p_\alpha \to \mathrm{VMOA}^\beta_p$ is bounded.

(ii) $\lim_{|z|\to 1}(1-|z|^2)^{\beta-1-\frac{2+\alpha}{p}}|\varphi(z)| = 0$.

(iii) $S_\varphi : A^p_\alpha \to \mathrm{VMOA}^\beta_p$ is compact.

Theorem 3.2 *Let $1 \le p < \infty, -1 < \alpha < \infty$ and $0 < \beta < \infty$. The following are equivalent.*

(i) $T_\varphi : A^p_\alpha \to \mathrm{VMOA}^\beta_p$ is bounded.

(ii) $\lim_{|z|\to 1}(1-|z|^2)^{\beta-\frac{2+\alpha}{p}}|\varphi'(z)| = 0$.

(iii) $T_\varphi : A^p_\alpha \to \mathrm{VMOA}^\beta_p$ is compact.

4 A special case

Take $\lambda(z) = \log \dfrac{1}{1-z}$. Then T_λ is a Cesàro operator. In [2], it is shown that the Cesàro operator T_λ is bounded from H^∞ to BMOA. We here consider the boundedness of operator T_φ acting from H^∞ to BMOA^β_p.

Theorem 4.1 *For $1 \le p < \infty$ and $0 < \beta < \infty$, $T_\varphi : H^\infty \to \mathrm{BMOA}^\beta_p$ is bounded if and only if $\varphi \in \mathrm{BMOA}^\beta_p$.*

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Fefferman's multiplier theorem and its recent developments - applications of the Besicovitch set to analytic problems

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1 Introduction

This is a brief note on the Fourier multiplier problems and its recent developments. There are no results mine.
For $f \in L^1(\mathbb{R}^n)$, we define the Fourier transform of f

$$\widehat{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{-2\pi i x \cdot \xi} d\xi.$$

Let us denote a unique extension of the Fourier transform on $L^2(\mathbb{R}^n)$ by $\mathscr{F}$. Then $\mathscr{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is an isometry and the Fourier inversion

$$\mathscr{F}^{-1}(f)(x) = \mathscr{F}(f)(-x)$$

holds on L^2.

Let $B_n(a,r) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$ $(a \in \mathbb{R}^n,\ r > 0)$. We consider the operator $S_{B_n(a,r)}$, defined for $f \in L^2(\mathbb{R}^n)$ by

$$S_{B_n(a,r)}(f) = \mathscr{F}^{-1} \chi_{B_n(a,r)} \mathscr{F}(f),$$

where $\chi_{B_n(a,r)}$ is the characteristic function of $B_n(a,r)$. We call this operator $S_{B_n(a,r)}$ Fourier multiplier for the ball $B_n(a,r)$. In particular, if $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then

$$S_{B_n(a,r)}(f)(x) = \int_{\mathbb{R}^n} \chi_{B_n(a,r)}(\xi) \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi.$$

The Fourier multiplier problem is then whether the Fourier multiplier $S_{B_n(0,1)}$ can be extended to a bounded linear operator from $L^p(\mathbb{R}^n)$ to itself.

2 Fefferman's multiplier theorem and the Besicovitch set

M. Riesz showed that $S_{B_1(0,1)}$ can be extended to a bounded linear operator on $L^p(\mathbb{R})$ for all $1 < p < \infty$. But C. Fefferman proved the following theorem in [2].

Theorem 2.1 (Fefferman, 1971)**.** *Suppose $n \geq 2$ and $1 < p < \infty$. The Fourier multiplier operator $S_{B_n(0,1)}$, initially defined on $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, can be extended to a bounded linear operator from $L^p(\mathbb{R}^n)$ to itself only for the case $p = 2$.*

It was surprising that Fefferman proved this theorem by using a construction of the Besicovitch set which could be used to give a solution to the Kakeya problem. The question was to find a set in $\mathbb{R}^2$ of the least area in which a segment of unit length could be moved so that it pointed in all possible directions.
Besicovitch showed a existence of a set yield a solution to the Kakeya problem.

Theorem 2.2. *There exists a set in $\mathbb{R}^2$ of Lebesgue measure zero which contains a unit segment in every direction. We call such set the Besicovitch set*

Idea of a construction of the Besicovitch set. Starting from the fixed triangle ABC, we subdivide the base AB into 2^N equal subintervals, with division points

$$A = A_0, A_1, ..., A_{2^N} = B.$$

Now we translate smaller triangles $A_{2j}A_{2j+2}C$ $(j = 0, ..., 2^N - 1)$ leftwards. Then we can incorporate each "blue areas" in Figure 1 into one triangle, which is similar to the original triangle ABC. This figure call $\Psi_1(ABC)$. So we carry out the above process on the small triangle with N replaced by $N-1$. We continue in this way, finally obtaining $\Psi_N(ABC)$. We can show that the area of $\Psi_N(ABC)$ is sufficiently small as large N and construct the Besicovitch set by using $\Psi_N(ABC)$.

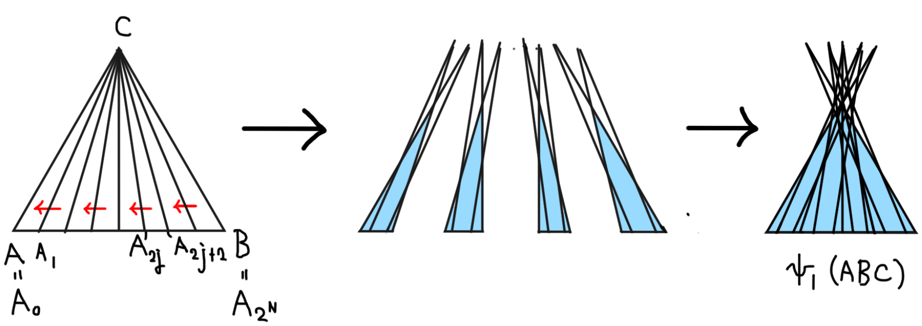

Figure 1

Outline of the proof of Fefferman's theorem. Note that it is enough to disprove L^p-boundedness of $S_{B_2(0,1)}$ on $L^p(\mathbb{R}^2)$ for $p < 2$. We assume that $S_{B_2(0,1)} : L^p(\mathbb{R}^2) \longrightarrow L^p(\mathbb{R}^2)$ is bounded.

The boundary of $B_2(0,1)$ has tangent lines in every directions. Then we can approximate any half-plane by suitable dilates and translates of $B_2(0,1)$. Using this approximation, we can get square function estimates for half-plane multiplier: For any collection of unit vectors $v_1, ..., v_k \in \mathbb{R}^2$ and any collection of functions $f_1, ..., f_k \in L^2(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$, there exists $C > 0$ such that

$$\left\| \left(\sum_{j=1}^{k} |S_{H_j}|^2 \right)^{\frac{1}{2}} \right\|_p \leq C \left\| \left(\sum_{j=1}^{k} |f_j|^2 \right)^{\frac{1}{2}} \right\|_p .$$

Here, $S_{H_j} = \mathscr{F}^{-1}\chi_{H_j}\mathscr{F}$ and $H_j = \{x \in \mathbb{R}^2 : x \cdot v_j > 0\}$.

We can exihibit a counterexample to square function estimates based on the construction of the Besicovitch set. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$, and 2^N rectangles $R_1, ..., R_{2^N}$, each having side length 1 and 2^{-N}, such that

(1) $\left|\bigcup_{j=1}^{2^N} R_j\right| < \varepsilon$,

(2) The $\widetilde{R}_j$ are pairwise disjoint, and $\left|\bigcup_{j=1}^{2^N} \widetilde{R}_j\right| = 1$.

Here $\widetilde{R}_j$ is the rectangle obtained by translating R_j two units along the longer side of R_j. We can use the Besicovitch set to construct $\{R_j\}$. (See Figure 2).

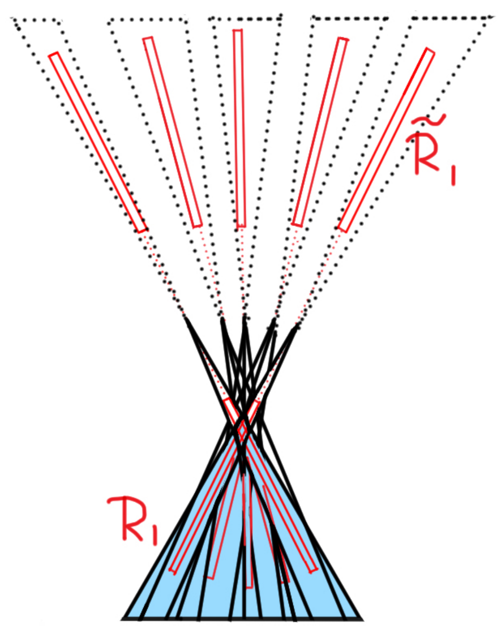

Figure 2

We set $f_j = \chi_{R_j}$, and let v_j be the unit vector which is parallel to the longer sides of R_j. By square function estimates, we have

$$\begin{aligned}
\left\| \left(\sum_{j=1}^{2^N} |S_{H_j}|^2 \right)^{\frac{1}{2}} \right\|_p &\leq C \left\| \left(\sum_{j=1}^{2^N} |\chi_{R_j}|^2 \right)^{\frac{1}{2}} \right\|_p \\
&\leq C \left[\int \left(\sum_{j=1}^{2^N} |\chi_{R_j}|^2 \right) dx \right]^{\frac{1}{2}} \left(\int_{\bigcup R_j} dx \right)^{1-\frac{p}{2}} \\
&= C\varepsilon^{1-\frac{p}{2}}.
\end{aligned}$$

On the other hands, we can see $C'\chi_{\widetilde{R}_j} \leq |S_{H_j}|$ ($\exists C' > 0$). Hence the result of this is then

$$C' \leq C\varepsilon^{1-\frac{p}{2}},$$

which is not possible if ε is sufficiently small. $\square$

3 Recent developments

Interest has arisen in studying analogues of the Fourier multiplier problem in the bilinear setting. Let $D \subset \mathbb{R}^{2d}$ be a domain. One may ask whether the bilinear Fourier multiplier

$$T_D(f,g)(x) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \chi_D(\xi,\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x\cdot(\xi+\eta)} d\xi d\eta$$

defined for Schwartz functions f, g on $\mathbb{R}^m$ extends to a bounded bilinear operator from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ for suitable ranges of p, q and r. Here χ_D denotes the characteristic function of D. For dimension $d = 1$, the case of $D = B_2(0,1) \subset \mathbb{R}^2$ was treated by Grafakos and Li in [4]. They showed the following theorem.

Theorem 3.1 (Grafakos and Li, 2006)**.** *Suppose* $2 \leq p, q < \infty$, $1 < r = \frac{pq}{p+q} \leq 2$. *Then* $T_{B_2(0,1)}$ *can be extended to a bounded bilinear operator from* $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ *to* $L^r(\mathbb{R})$.

For $d \geq 2$ and $B_{2d}(0,1) \subset \mathbb{R}^{2d}$, the following theorem proved by Diestel and Grafakos in [1].

Theorem 3.2 (Diestel and Grafakos, 2007)**.** *Let* $m \geq 2$ *and* $1/p + 1/q = 1/r$ *with exactly one of* p, q, $r(r-1)$ *strictly less than* 2. *Then* $T_{B_{2m}(0,1)}$ *is not extendable to a bounded bilinear operator from* $L^p(\mathbb{R}^m) \times L^q(\mathbb{R}^m)$ *to* $L^r(\mathbb{R}^m)$.

In [5], Grafakos and Reguera generalized this result to replace the ball $B_{2d}(0,1)$ with a domain D which have a certain property.

Theorem 3.3 (Grafakos and Reguera, 2010)**.** *Let* $m \geq 2$ *and* $1/p+1/q = 1/r$ *with at least one of* p, q, $r(r-1)$ *strictly less than* 2. *If* D *is a compact, strictly convex domain which* ∂D *is a smooth hypersurface, then* T_D *is not extendable to a bounded bilinear operator from* $L^p(\mathbb{R}^m) \times L^q(\mathbb{R}^m)$ *to* $L^r(\mathbb{R}^m)$.

Moreover, Gautam obtained the following generalization of Theorem 3.3 for $d = 2$ in [3].

Theorem 3.4 (Gautam, 2012)**.** *Suppose* $1/p+1/q = 1/r$ *with exactly one of* p, q, $r(r-1)$ *strictly less than* 2. *Let* $D \in \mathbb{R}^4$ *which* ∂D *is smooth in some neighborhood* $U \subset \mathbb{R}^4$, *and suppose that either* D *or* $\mathbb{R}^4 \backslash D$ *is strictly convex in* U. *Then* T_D *is not extendable to a bounded bilinear operator from* $L^p(\mathbb{R}^2) \times L^q(\mathbb{R}^2)$ *to* $L^r(\mathbb{R}^2)$.

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Point multipliers on Banach modules

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This is an introduction of a paper [2] by Ghodrat and Sady.

Background

Let A be a Banach algebra with identity 1_A. We denote by $\sigma(a)$ the spectrum of $a \in A$. The Gleason [3] and Kahane and Żelazko [4] have proven independently the following theorem known as the Gleason-Kahane-Żelazko theorem:

Theorem. *The following conditions are equivalent for each linear functional φ on A:*
(a) *φ is a non-zero and multiplicative.*
(b) *$\varphi(1_A) = 1$ and $\varphi(a) \neq 0$ for every invertible element $a \in A$.*
(c) *$\varphi(a) \in \sigma(a)$ for every $a \in A$.*

The Gleason-Kahane-Żelazko theorem is a theorem on linear preserver problem. (a)$\Leftrightarrow$(b) shows that the unital linear maps which preserves invertibility in one direction are precisely the non-zero multiplicative linear functionals. Motivated by the theorem, Kaplansky [5] raised the following question.

Kaplansky's Problem. *Let A, B be unital semisimple Banach algebras and let $T : A \to B$ be a surjective linear transform which satisfies $T(1) = 1$ and T preserves invertibility in one direction, i.e., $T(a)$ is invertible whenever $a \in A$ is invertible. Is it true that T necessarily satisfies $T(a^2) = T(a)^2$?*

Note that for a linear transform $T : A \to B$ with $T(1) = 1$, T preserves invertibility in one direction if and only if $\sigma(T(a)) \subset \sigma(a)$ for each $a \in A$. The problem is still open, however, it follows from the Gleason-Kahane-Żelazko theorem that the problem is affirmative for unital semisimple commutative Banach algebras, more precisely, such T can be represented by a composition operator on the Gelfand transform:

$$\widehat{T(a)} = \hat{a} \circ \tau,$$

where τ is a continuous mapping from the maximal ideal space of B into the maximal ideal space of A.

In this proceeding, we present some generalizations of the Gleason-Kahane-Żelazko theorem given by Ghodrat and Sady [2].

Point multipliers

Let A be a Banach algebra. The set of all non-zero multiplicative linear functional on A is denoted by $\sigma(A)$. A *Banach left A-module* is a Banach space $\mathcal{X}$ which is also A-module and satisfies

$$\|a \cdot x\| \leq \|a\|\|x\|$$

for every $a \in A$ and $x \in \mathcal{X}$. If, in addition, A has identity 1 and $1 \cdot x = x$ for each $x \in \mathcal{X}$, this Banach left A-module is called *unital*.

Definition. Let $\varphi \in \sigma(A) \cup \{0\}$. A bounded linear functional ξ on a Banach left A-module $\mathcal{X}$ is called a *point multiplier at φ* if

$$\xi(a \cdot x) = \varphi(a)\xi(x)$$

for every $a \in A$ and $x \in \mathcal{X}$. The set of all non-zero point multipliers ξ at some $\varphi \in \sigma(A) \cup \{0\}$ which satisfies $\|\xi\| \leq 1$ is denoted by $\sigma_A(\mathcal{X})$.

Note that a point multiplier at φ is a continuous A-homomorphism from $\mathcal{X}$ into $\mathbb{C}$, if A-module operation on $\mathbb{C}$ is defined by

$$a \cdot z := \varphi(a)z$$

for every $a \in A$ and $z \in \mathbb{C}$. Conversely, every Banach A-module operation on $\mathbb{C}$ is represented as above, and thus every continuous A-homomorphism from $\mathcal{X}$ into $\mathbb{C}$ is a point multiplier.

We denote by $\Delta_A(X)$ the set of all closed submodule P of $\mathcal{X}$ of codimension 1. It is easy to see that the kernel of $\xi \in \sigma_A(\mathcal{X})$ belongs to $\Delta_A(\mathcal{X})$. Conversely, each $P \in \Delta_A(\mathcal{X})$ is the kernel of some $\xi \in \sigma_A(\mathcal{X})$. Note that the map

$$\sigma_A(\mathcal{X}) \ni \xi \mapsto \ker(\xi) \in \Delta_A(\mathcal{X})$$

is NOT injective because, if $\xi \in \sigma_A(\mathcal{X})$ and $0 < |\lambda| \leq 1$, then $\lambda\xi \in \sigma_A(\mathcal{X})$.

Ghodrat and Sady obtained a generalization of (a)$\Leftrightarrow$(b) of the Gleason-Kahane-Żelazko theorem as follow.

Theorem ([2, Theorem 3.1])**.** *Let A be a unital Banach algebra and let $\mathcal{X}$ be a unital Banach left A-module. Then we have the following.*
(i) *Let ξ be a linear functional. In order that ξ satisfies*

$$\xi(a \cdot x) = \varphi(a)\xi(x)$$

for every $a \in A$ and $x \in \mathcal{X}$, it is necessary and sufficient that its kernel $\ker(\xi)$ *is submodule of $\mathcal{X}$.*
(ii) *Let ξ is a non-zero bounded linear functional. In order that ξ is a point multiplier on $\mathcal{X}$, it is necessary and sufficient that*

$$\xi(a \cdot x) \neq 0$$

for every invertible element a of A and $x \in \mathcal{X} \setminus \ker(\xi)$.

Spectra of elements in Banach modules

Let A be a Banach algebra and let $\mathcal{X}$ be a Banach left A-module. Ghodrat and Sady introduce a spectra of elements in Banach module as follow.

Definition ([2, Definition 3.10])**.** Let $\mathcal{F} \subset \sigma_A(\mathcal{X})$. For each $x \in \mathcal{X}$, we set

$$\sigma_h^{\mathcal{F}}(x) := \{\xi(x) : \xi \in \mathcal{F}\}.$$

We also set $\sigma_h(x) := \sigma_h^{\sigma_A(\mathcal{X})}(x)$.

Each unital commutative Banach algebra A can be regarded as a Banach A-module. Then $\sigma(A) \subset \sigma_A(A)$. If we set $\mathcal{F} := \sigma(A)$, the spectrum $\sigma_h^{\mathcal{F}}(x)$ of $x \in A$ as an element of Banach module coincides with the usual spectrum $\sigma(x)$.

For a compact Hausdorff space X and a Banach space E, we will denote the Banach space of all continuous functions on X with values in E by $C(X, E)$. With pointwise operation

$$(f \cdot F)(x) := f(x)F(x) \ \ (f \in C(X), F \in C(X, E), x \in X),$$

$C(X, E)$ is a Banach $C(X)$-module. Since $C(X, E)$ is the injective tensor product of $C(X)$ and E, we see that

$$\sigma_{C(X)}(C(X, E)) = \{\Lambda \circ \delta_x : x \in X, \Lambda \in (E^*)_1 \setminus \{0\}\},$$

where δ_x is the point evaluation at $x \in X$ and $(E^*)_1$ is the unit all of the dual space of E. Thus the spectrum of $F \in C(X, E)$ is a subset of

$$\{\Lambda(F(x)) : x \in X, \Lambda\}.$$

Ghodrat and Sady also characterized maps which preserve spectrum. To state the theorem, we need some notations. Let A be a Banach algebra, and let $\mathcal{X}$ be a Banach left A-module. For $x \in \mathcal{X}$, define a function $\hat{x} : \Delta_A(\mathcal{X}) \to \bigcup_{P \in \Delta_A(\mathcal{X})} \mathcal{X}/P$ by

$$\hat{x}(P) := x + P \ \ (P \in \Delta_A(\mathcal{X})).$$

For a subset S of a vector space, the convex hull is denoted by $\mathrm{co}(S)$.

Theorem ([2, Theorem 3.12])**.** *Let A be a unital Banach algebra, let $\mathcal{X}$ and $\mathcal{Y}$ be unital left Banach A-modules, and let $\mathcal{F}$ and $\mathcal{F}'$ be weak $*$-compact subset of $\sigma_A(\mathcal{X})$ and $\sigma_A(\mathcal{Y})$, respective, such that $\bigcap_{\eta \in \mathcal{F}} \ker(\eta) = \{0\}$ and $\bigcap_{\xi \in \mathcal{F}'} \ker(\xi) = \{0\}$. Suppose that $T : \mathcal{X} \to \mathcal{Y}$ is a surjective bounded linear operator which satisfies*

$$\sigma_h^{\mathcal{F}'}(T(x)) = \sigma_h^{\mathcal{F}}(x)$$

for every $x \in \mathcal{X}$. Then there are subsets $E_0 \subset \Delta_A(X)$ and $F_0 \subset \Delta_A(Y)$ which satisfies $\sigma_A(\mathcal{X}) \subset \mathrm{co}\{\eta \in \mathcal{X}^* : \ker(\eta) \in E_0\}$ and $\sigma_A(\mathcal{Y}) \subset \mathrm{co}\{\xi \in \mathcal{Y}^* : \ker(\xi) \in F_0\}$, and a bijection $h : F_0 \to E_0$ such that

$$\widehat{T(x)}(P) = J_P(\widehat{x}(h(P))) \ \ (\forall x \in \mathcal{X}, \forall P \in \Delta_A(\mathcal{Y}))$$

where $J_P : \mathcal{X}/h(P) \to \mathcal{Y}/P$ is a bijective linear map for each $P \in F_0$.

We present outline of the proof.

Note that T is injective. Indeed, if $T(x)=0$, then

$$\sigma_h^{\mathcal{F}}(x)=\sigma_h^{\mathcal{F}'}(T(x))=\{0\},$$

and since $\bigcap_{\eta\in\mathcal{F}} \ker(\eta)=\{0\}$, it follows that $x=0$.

Consider the spectral states

$$S_{\mathcal{F}}(\mathcal{X}):=\{\eta\in\mathcal{Y}^*:\eta(x)\in\mathrm{co}(\sigma_h^{\mathcal{F}}(x))\ \ (\forall x\in\mathcal{X})\}$$
$$S_{\mathcal{F}'}(\mathcal{Y}):=\{\xi\in\mathcal{X}^*:\xi(y)\in\mathrm{co}(\sigma_h^{\mathcal{F}'}(y))\ \ (\forall y\in\mathcal{Y})\}.$$

Then $S_{\mathcal{F}}(\mathcal{X})$ and $S_{\mathcal{F}'}(\mathcal{Y})$ are convex set. Applying a similar argument as in [1, Lemma 4.1.16], we can prove that the extreme points of $S_{\mathcal{F}}(\mathcal{X})$ and $S_{\mathcal{F}'}(\mathcal{Y})$ are contained in $\mathcal{F}$ and $\mathcal{F}'$, respectively.

Since T preserves the spectrum, we see that the adjoint operator T^* preserves the extreme points of the spectral states:

$$T^*(\mathrm{ext}(S_{\mathcal{F}'}(\mathcal{Y})))=\mathrm{ext}(S_{\mathcal{F}}(\mathcal{X})).$$

Thus for each $\xi\in S_{\mathcal{F}'}(\mathcal{Y})$, the functional $T^*(\xi)=\xi\circ T$ is a point multiplier on $\mathcal{X}$ at some point.

Let

$$E_0:=\{\ker(\eta):\eta\in\mathrm{ext}(S_{\mathcal{F}}(\mathcal{X}))\},$$
$$F_0:=\{\ker(\xi):\xi\in\mathrm{ext}(S_{\mathcal{F}'}(\mathcal{Y}).$$

Then E_0 and F_0 are subsets of $\Delta_A(\mathcal{X})$ and $\Delta_A(\mathcal{Y})$, respectively. Define $h:F_0\to E_0$ by the following manner: for each $P\in F_0$, choose $\xi\in\mathrm{ext}(S_{\mathcal{F}}(\mathcal{X}))$ so that $P=\ker(\xi)$ and put

$$h(P):=\ker(\xi\circ T).$$

Then we see that h is a well-defined bijection.

As in elementary algebra, for each $P\in F_0$, the map $J_P:\mathcal{X}/h(P)\to\mathcal{Y}/P$ defined by

$$J_P(x+h(P)):=T(x)+P$$

is a well-defined bijective linear map. Thus we have

$$\widehat{T(x)}(P)=T(x)+P=J_P(x+h(P))=J_P(\widehat{x}(h(P)))$$

for each $P\in F_0$ and $x\in\mathcal{X}$.

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Surjective isometries on a Lipschitz space of analytic functions on the open unit disc
単位開円板上の正則関数のなすリプシッツ空間とその上の全射等距離写像

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1　導入

ノルム空間 $(N, \|\cdot\|_N)$ 上で定義された写像 S が

$$\|S(f) - S(g)\|_N = \|f - g\|_N \qquad (\forall f, g \in N)$$

をみたすとき，S を等距離写像という．$\mathbb{D}$ を複素平面の単位開円板とし $H(\mathbb{D})$ を $\mathbb{D}$ 上の正則関数全体のなす複素線型空間とする．ハーディー空間

$$H^p = \left\{ f \in H(\mathbb{D}) : \|f\|_p = \sup_{0<r<1} \left[\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right]^{1/p} < \infty \right\} \qquad (1 \leq p < \infty)$$

及び $H^\infty = \left\{ f \in H(\mathbb{D}) : \|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)| < \infty \right\}$ 上の複素線形等距離写像は 1960 年代に解明されている．

定理 (deLeeuw, Rudin and Wermer [3]).　*1.* S が $(H^\infty, \|\cdot\|_\infty)$ 上の全射複素線形等距離写像であるための必要十分条件は，

$$S(f)(z) = \alpha f(\phi(z)) \qquad (\forall f \in H^\infty, z \in \mathbb{D})$$

となる $\alpha \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ と等角写像 $\phi\colon \mathbb{D} \to \mathbb{D}$ が存在することである．

2. S が $(H^1, \|\cdot\|_1)$ 上の全射複素線形等距離写像であるための必要十分条件は，

$$S(f)(z) = \alpha\phi'(z) f(\phi(z)) \qquad (\forall f \in H^1, z \in \mathbb{D})$$

となる $\alpha \in \mathbb{T}$ と等角写像 $\phi\colon \mathbb{D} \to \mathbb{D}$ が存在することである．

1959年にNagasawa [15]は，関数環上の全射複素線形等距離写像の構造を解明している．deLeeuw, Rudin and Wermer [3]の H^∞ に対する結果は，Nagasawaの定理の特別な場合ということが出来る．

Forelliは H^p 上の，全射とは限らない，複素線形等距離写像を決定している．ここでは特に全射の場合の結果について言及する．

定理 (Forelli, [6])**.** p を $1 \leq p < \infty$ *and* $p \neq 2$ をみたす実数とする．S が $(H^p, \|\cdot\|_p)$ 上の全射複素線形等距離写像であるための必要十分条件は，

$$S(f)(z) = \alpha(\phi'(z))^{1/p} f(\phi(z)) \qquad (\forall f \in H^p, z \in \mathbb{D})$$

となる $\alpha \in \mathbb{T}$ と等角写像 $\phi\colon \mathbb{D} \to \mathbb{D}$ が存在することである．

ハーディー空間 H^p 上の全射複素線形等距離写像はこのように決定されているが，特に H^∞ 上の全射とは限らない複素線形等距離写像の構造が解明されているのかを筆者は知らない．ハーディー空間とは限らない，正則関数のなすバナッハ空間上の複素線形等距離写像は様々な空間において研究がなされている（たとえば [1, 2, 5, 7, 9, 10, 13] を参照されたい）．

Novinger and Oberlin [16] は，ハーディー空間 H^p に関連した $H(\mathbb{D})$ の部分空間

$$\mathcal{S}^p = \{f \in H(\mathbb{D}) : f' \in H^p\}$$

に次の2種類のノルムを与え，それぞれのバナッハ空間に対する全射とは限らない複素線形等距離写像の形を決定した．

$$\|f\|_\sigma = |f(0)| + \|f'\|_p, \quad \|f\|_\Sigma = \|f\|_\infty + \|f'\|_p \qquad (f \in \mathcal{S}^p).$$

ただし $f' \in H^p$ ならば f は $\mathbb{D}$ の閉包 $\bar{\mathbb{D}}$ 上に連続的に拡張可能であるから（たとえばDuren [4, Theorem 3.11] 参照），$\|f\|_\infty$ は意味をもつ．ここでもNovinger and Oberlinの結果の全射の場合について述べることにする．

定理 (Novinger and Oberlin [16])**.** p を $1 \leq p < \infty$ *and* $p \neq 2$ をみたす実数とする．

1. S が $(\mathcal{S}^p, \|\cdot\|_\sigma)$ 上の全射複素線形等距離写像であるための必要十分条件は

$$S(f)(z) = cf(0) + \int_{[0,z]} (\phi'(\zeta))^{1/p} f'(\phi(\zeta))\, d\zeta \qquad (\forall f \in \mathcal{S}^p, z \in \mathbb{D})$$

をみたす $c \in \mathbb{T}$ 及び等角写像 $\phi\colon \mathbb{D} \to \mathbb{D}$ が存在することである．

2. S が $(\mathcal{S}^p, \|\cdot\|_\Sigma)$ 上の全射複素線形等距離写像であるための必要十分条件は

$$S(f)(z) = cf(\phi(z)) \qquad (\forall f \in \mathcal{S}^p, z \in \mathbb{D})$$

をみたす $c \in \mathbb{T}$ 及び等角写像 $\phi\colon \mathbb{D} \to \mathbb{D}$ が存在することである．

2 主定理

Novinger and Oberlin の定理では，$p=\infty$ を除く $\mathcal{S}^p$ に対する全射複素線形等距離写像の構造を解明している．それでは $\mathcal{S}^\infty = \{f \in H(\mathbb{D}) : f' \in H^\infty\}$ 上の全射複素線形等距離写像はどのような形をしているのであろうか．筆者はこれまでに保存問題の視点から全射等距離写像を調べてきた．そこで $\mathcal{S}^\infty$ のノルム $\|f\|_\sigma = |f(0)| + \|f'\|_\infty$ 及び $\|f\|_\Sigma = \|f\|_\infty + \|f'\|_\infty$ に関する全射等距離写像を考察し，その構造を明らかにした．以下が主定理である．

定理 1. S が $(\mathcal{S}^\infty, \|\cdot\|_\sigma)$ 上の全射等距離写像であるための必要十分条件は，$c_0, c_1, \lambda \in \mathbb{T}$ 及び $a \in \mathbb{D}$ が存在して

$$
\begin{aligned}
S(f)(z) &= S(0)(z) + c_0 f(0) + \int_{[0,z]} c_1 f'\left(\lambda \frac{z-a}{1-\overline{a}\zeta}\right) d\zeta && (\forall f \in \mathcal{S}^p, z \in \mathbb{D}) \quad or \\
S(f)(z) &= S(0)(z) + c_0 \overline{f(0)} + \int_{[0,z]} c_1 f'\left(\lambda \frac{z-a}{1-\overline{a}\zeta}\right) d\zeta && (\forall f \in \mathcal{S}^p, z \in \mathbb{D}) \quad or \\
S(f)(z) &= S(0)(z) + c_0 f(0) + \int_{[0,z]} c_1 \overline{f'\left(\overline{\lambda \frac{z-a}{1-\overline{a}\zeta}}\right)} d\zeta && (\forall f \in \mathcal{S}^p, z \in \mathbb{D}) \quad or \\
S(f)(z) &= S(0)(z) + c_0 \overline{f(0)} + \int_{[0,z]} c_1 \overline{f'\left(\overline{\lambda \frac{z-a}{1-\overline{a}\zeta}}\right)} d\zeta && (\forall f \in \mathcal{S}^p, z \in \mathbb{D})
\end{aligned}
$$

が成り立つことである．

定理 2. S が $(\mathcal{S}^\infty, \|\cdot\|_\Sigma)$ 上の全射等距離写像であるための必要十分条件は，$c, \lambda \in \mathbb{T}$ が存在して

$$
\begin{aligned}
S(f)(z) &= S(0)(z) + cf(\lambda z) && (\forall f \in \mathcal{S}^p, z \in \mathbb{D}) \quad or \\
S(f)(z) &= S(0)(z) + \overline{cf(\overline{\lambda z})} && (\forall f \in \mathcal{S}^p, z \in \mathbb{D})
\end{aligned}
$$

が成り立つことである．

証明の概略. 定理 1, 2 の証明のアイディアは本質的に同じであるので，ここでは定理 2 の証明の概略を述べることとする．詳細は [14] をご覧いただきたい．

まず Mazur-Ulam の定理 [11] より，$S - S(0)$ は全射実線形等距離写像となる．Mazur-Ulam の定理の簡潔な照明は Väisälä [17] によって与えられている．$f' \in H^\infty$ のゲルファント変換を $\widehat{f'}$ で表し，∂_{H^∞} を H^∞ のシロフ境界とする．$f \in \mathcal{S}^\infty$ は $\bar{\mathbb{D}}$ 上に連続的に拡張可能であるから，その一意的拡張を $\hat{f}$ で表す．この記号の用法により混乱は生じないものと思われる．このとき次が成り立つ．

$$
\|f\|_\Sigma = \sup_{z\in\mathbb{D}} |f(z)| + \sup_{\zeta\in\mathbb{D}} |f'(\zeta)| = \sup_{z\in\mathbb{T}} |\hat{f}(z)| + \sup_{\zeta\in\partial_{H^\infty}} |\widehat{f'}(\zeta)| = \sup_{(z,w,\zeta)\in\mathbb{T}^2\times\partial_{H^\infty}} |\hat{f}(z) + w\widehat{f'}(\zeta)|.
$$

そこで $U\colon \mathcal{S}^\infty \to C(\mathbb{T}^2 \times \partial_{H^\infty})$ を

$$
U(f)(z,w,\zeta) = \hat{f}(z) + w\widehat{f'}(\zeta) \qquad (\forall f \in \mathcal{S}^\infty, (z,w,\zeta) \in \mathbb{T}^2 \times \partial_{H^\infty})
$$

により定め，$B = U(\mathcal{S}^\infty)$ とおけば，U を $\mathcal{S}^\infty$ から B への全射複素線形等距離写像とみなせる．$V = USU^{-1}$ とおけば，V は B 上の全射実線形等距離写像となる．$\mathrm{ext}(B_1^*)$ を B の双対空間 B^* の

閉単位球 B_1^* の端点全体の集合とする．Banach-Stone の定理の証明で用いられる手法に端点を決定する方法が知られているが，実線形等距離写像に対してもそれと類似の手法を適用することができる．筆者には $\mathrm{ext}(B_1^*)$ を完全に決定することは出来ていないが，筑波大学の川村一宏先生のアイディアを用いることで次を示すことが出来た．

$$V_*(\{\lambda\delta_x : \lambda \in \mathbb{T}, x \in \mathbb{T}^2 \times \partial_{H^\infty}\}) = \{\lambda\delta_x : \lambda \in \mathbb{T}, x \in \mathbb{T}^2 \times \partial_{H^\infty}\} \tag{1}$$

ただし $V_*\colon B^* \to B^*$ は

$$V_*(\eta)(a) = \mathrm{Re}\,\eta(V(a)) - i\mathrm{Re}\,\eta(V(ia)) \qquad (\forall\eta \in B^*, a \in B)$$

により定められる全射実線形等距離写像であり，$\delta_x\colon B \to \mathbb{C}$ は $\delta_x(a) = a(x)\,(a \in B)$ により定まる点値汎関数である．(1) により V は荷重合成作用素とその複素共役により表示されることがわかる．$S = U^{-1}VU$ より S の形は定まるが，そこには変換 U を用いることにより導入された変数 w, ζ などが含まれる．これらは S の形には本来影響を及ぼさないはずであるから，これらの変数を除去する必要がある．実際それが可能であり，その結果上記の表示を得る．

定理 2 の表示を得るために，関数環上の全射実線形等距離写像の構造定理 [8, 12] を用いている．□

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Algebraic structures for means

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1 Abstract

Some means on the positive matrices can be represented by algebraic midpoint.

2 Strictly positive matrices

Let $\mathbb{M}_n(\mathbb{C})$ be the space of $n \times n$ matrices with complex entries. We say that $A \in \mathbb{M}_n(\mathbb{C})$ is positive if

$$\langle \boldsymbol{x}, A\boldsymbol{x} \rangle \geq 0 \quad \text{for all } \boldsymbol{x} \in \mathbb{C}^n,$$

and strictly positive if, in addition,

$$\langle \boldsymbol{x}, A\boldsymbol{x} \rangle > 0 \quad \text{for all } \boldsymbol{x} \neq 0.$$

We denote by $\mathbb{P}_n$ the set of $n \times n$ strictly positive matrices. $\mathbb{P}_n$ is not a linear subspace of $M_n(\mathbb{C})$, but a convex cone. For $A, B \in \mathbb{P}_n$, we use the notation $A \geq B$ to mean that the matrix $A - B$ is positive. In particular, $\mathbb{P}_1 = \mathbb{R}_+$ is the set of all positive real numbers.

3 Binary operations

A magma $(S, \circ)$ is a set S with a binary operation $\circ : S \times S \to S$, $(a, b) \mapsto a \circ b$ for any $a, b \in S$. An automorphism ϕ of a magma $(S, \circ)$ is a bijection $\phi : S \to S$ which preserves the magma operation, that is $\phi(a \circ b) = \phi(a) \circ \phi(b)$ for any $a, b \in S$. If there exists an element $e \in (S, \circ)$ such that $e \circ a = a \circ e = a$ for any $a \in S$, then e is called the identity of $(S, \circ)$. Let $(S, \circ)$ has the identity. For $a \in (S, \circ)$, if there exists an element $a' \in S$ such that $a \circ a' = a' \circ a = e$, then a' is called an inverse of a.

Definition 1. Let $(S, \oplus)$ be a magma.

- We say that $(S, \circ)$ is associative if $(a \circ b) \circ c = a \circ (b \circ c)$ for any $a, b, c \in S$.
- We say that $(S, \circ)$ is left-cancellative if $a \circ b = a \circ c$ implies $b = c$ for any $a, b, c \in S$.

- We say that $(S, \circ)$ is right-cancellative if $b \circ a = c \circ a$ implies $b = c$ for any $a, b, c \in S$.
- We say that $(S, \circ)$ is commutative if $a \circ b = b \circ a$ for any $a, b \in S$.
- We say that $(S, \circ)$ is uniquely 2-divisible if, for any $a \in S$ there exists a unique element $b \in S$ such that $a = b \circ b$. The element b is called the half of a.

In this paper, we often use the symbol $\oplus$ for a binary operation. For uniquely 2-divisible magma $(S, \oplus)$, we denote by $\frac{1}{2} \otimes a$ the half of $a \in S$.

3.1 Semi-group midpoints

An associative magma is called a semi-group. In this paper, for consistency, we use the term "semi-group midpoint".

Definition 2. Let $(S, \oplus)$ be a uniquely 2-divisible commutative semi-group and $a, b \in S$. We call $\frac{1}{2} \otimes (a \oplus b)$ the semi-group midpoint of a and b.

3.2 Gyromidpoints

Definition 3. A magma $(G, \oplus)$ is called a gyrogroup if it satisfies the following (G1) to (G5).

(G1) $(G, \oplus)$ has the identity e.

(G2) For any $a \in (G, \oplus)$, a has the inverse $\ominus a$.

(G3) For any $a, b, c \in G$, there exists a unique element $\mathrm{gyr}[a, b]c$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c.$$

(G4) For any $a, b \in G$, the map $\mathrm{gyr}[a, b] : G \to G$ defined by $c \mapsto \mathrm{gyr}[a, b]c$ for any c is an automorphism of the magma $(G, \oplus)$.

(G5) For any $a, b \in G$, $\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a, b]$.

A gyrogroup $(G, \oplus)$ is gyrocommutative if the following (G6) is satisfied.

(G6) For any $a, b \in G$, $a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$.

An algebraic midpoint for a gyrogroup is defined as follows.

Definition 4. Let $(X, \oplus)$ be a uniquely 2-divisible gyrocommutative gyrogroup, and $a, b \in G$. The element

$$\frac{1}{2} \otimes (a \oplus \mathrm{gyr}[a, \ominus b]b)$$

is called gyromidpoint of a and b.

Let $(X, \oplus)$ be a commutative group. Then $(X, \oplus)$ is both a commutative semi-group and gyrocommutaive gyrogroup. In this case, if $(X, \oplus)$ is uniquely 2-divisible, then algebraic midpoint as group and as gyrogroup correspond.

4 Algebraic structures like a linear space

4.1 Gyro linear spaces

It is known that several gyrocommutative gyrogroup have the structure like a linear space in the sense of following definition.

Definition 5. Let $(X, \oplus)$ be a gyrocommutative gyrogroup. Let $\otimes$ be a map $\otimes : \mathbb{R} \times X \to X$. We say that $(X, \oplus, \otimes)$ is a gyrolinear space if the following conditions (GL1) to (GL5) are fulfilled.

(GL1) $1 \otimes a = a$ for any $a \in G$.

(GL2) $(\lambda + \mu) \otimes a = (\lambda \otimes a) \oplus (\mu \otimes a)$ for any $\lambda, \mu \in \mathbb{R}$ and $a \in X$.

(GL3) $(\lambda\mu) \otimes a = \lambda \otimes (\mu \otimes a)$ for any $\lambda, \mu \in \mathbb{R}$ and $a \in X$.

(GL4) $\mathrm{gyr}[u, v](\lambda \otimes a) = \lambda \otimes \mathrm{gyr}[u, v]a$ for any $\lambda \in \mathbb{R}$ and $u, v, a \in X$.

(GL5) $\mathrm{gyr}[\lambda \otimes u, \mu \otimes u] = id_X$ for any $\lambda, \mu \in \mathbb{R}$ と $u \in X$.

The map $\otimes : \mathbb{R} \times X \to X$ is called scalar multiplication.

If $(X, \oplus, \otimes)$ is a gyrolinear space, it is easy to check that $(X, \oplus)$ is uniquely 2-divisible. In this case, for $a \in X$, the notation $\frac{1}{2} \otimes a$ has two meaning. One is the half element of a, and the other is scalar multiplication. However, these two are the same element.

4.2 A structure like a linear space for semi-group

We consider a structure like a linear space for semi-group. In this paper, we use the term "semi-linear space".

Definition 6. Let $(X, \oplus)$ be a commutative semi group. Let $\otimes$ be a map $\otimes : \mathbb{R}_+ \times X \to X$. We say that $(X, \oplus, \otimes)$ is a semi-linear space if the following conditions (SL1) to (SL4) are fulfilled.

(SL1) $1 \otimes a = a$ for any $a \in X$.

(SL2) $(\lambda + \mu) \otimes a = (\lambda \otimes a) \oplus (\mu \otimes a)$ for any $\lambda, \mu \in \mathbb{R}_+$ and $a \in X$.

(SL3) $\lambda \otimes (a \oplus b) = \lambda \otimes a \oplus \lambda \otimes b$ for any $\lambda \in \mathbb{R}_+$ and $a, b \in X$.

(SL4) $(\lambda\mu) \otimes a = \lambda \otimes (\mu \otimes a)$ for any $\lambda, \mu \in \mathbb{R}_+$ and $a \in X$.

We call the map $\otimes$ scalar multiplication.

If $(X, \oplus, \otimes)$ is a semi-linear space, it is easy to check that $(X, \oplus)$ is uniquely 2-divisible. In this case, for $a \in X$, the notation $\frac{1}{2} \otimes a$ has two meaning. One is the half element of a, and the other is scalar multiplication. However, these two are the same element.

5 Means on $\mathbb{P}_n$

The map $M : \mathbb{P}_n \times \mathbb{P}_n \to \mathbb{P}_n$ is called a mean, if the following conditions (M1) to (M5) are fulfilled.

(M1) If $a \leq b$, then $a \leq M(a,b) \leq b$.

(M2) $M(a,b) = M(b,a)$.

(M3) $M(a,b)$ is monotone increasing in a, b.

(M4) $M(x^*ax, x^*bx) = x^*M(a,b)x$ for all $a, b \in \mathbb{P}_n$ and nonsingular $x \in \mathbb{M}_n(\mathbb{C})$.

(M5) $M(a,b)$ is continuous in a, b.

5.1 Examples

Example 7. Define the binary operation $\oplus_A$ on $\mathbb{P}_n$ by $\oplus_A = +$, that is,

$$a \oplus_A b = a + b \quad \text{for all } a, b \in \mathbb{P}_n,$$

then $(\mathbb{P}_n, \oplus_A)$ is a uniquely 2-divisible commutative semi-group. Denote by $A(a,b)$ the semi-group midpoint of a and b, that is,

$$A(a,b) = \frac{1}{2} \otimes_A (a \oplus_A b) = \frac{A+B}{2}$$

then $A(a,b)$ is the arithmetic mean of a and b. Moreover, define the map $\otimes_A : \mathbb{R}_+ \times \mathbb{P}_n \to \mathbb{P}_n$ by

$$\lambda \otimes_A a = \lambda a \quad \text{for all } a \in \mathbb{P}_n \text{ and } \lambda \in \mathbb{R}_+,$$

then $(\mathbb{P}_n, \oplus_A, \otimes_A)$ is a semi-linear space. Clearly, it is a cone of $\mathbb{M}_n(\mathbb{C})$.

Example 8. Define the binary operation $\oplus_H$ on $\mathbb{P}_n$ by

$$a \oplus_H b = (a^{-1} + b^{-1})^{-1} \quad \text{for all } a, b \in \mathbb{P}_n,$$

then $(\mathbb{P}_n, \oplus_H)$ is a uniquely 2-divisible commutative semi-group. Denote by $H(a,b)$ the semi-group midpoint of a and b, that is,

$$H(a,b) = \frac{1}{2} \otimes_H (a \oplus_H b) = 2(a^{-1} + b^{-1})^{-1},$$

then $H(a,b)$ is the harmonic mean of a and b. Moreover, define the map $\otimes_H : \mathbb{R}_+ \times \mathbb{P}_n \to \mathbb{P}_n$ by

$$\lambda \otimes_H a = \frac{1}{\lambda} a \quad \text{for all } a \in \mathbb{P}_n \text{ and } \lambda \in \mathbb{R}_+,$$

then $(\mathbb{P}_n, \oplus_H, \otimes_H)$ is a semi-linear space.

Example 9. Define the binary operation $\oplus_G$ on $\mathbb{P}_n$ by

$$a \oplus_G b = a^{\frac{1}{2}} b a^{\frac{1}{2}} \quad \text{for all } a, b \in \mathbb{P}_n,$$

then $(\mathbb{P}_n, \oplus_G)$ is a uniquely 2-divisible gyrocommutative gyrogroup. Denote by $G(a,b)$ the gyromidpoint of a and b, that is,

$$G(a,b) = \frac{1}{2} \otimes_G (a \boxplus_G b) = a^{\frac{1}{2}}(a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}})^{-\frac{1}{2}} a^{\frac{1}{2}},$$

then $G(a,b)$ is the geometric mean of a and b. Moreover, define the map $\otimes_G : \mathbb{R} \times \mathbb{P}_n \to \mathbb{P}_n$ by

$$\lambda \otimes_G a = a^{\lambda} \quad \text{for all } a \in \mathbb{P}_n \text{ and } \lambda \in \mathbb{R},$$

then $(\mathbb{P}_n, \oplus_G, \otimes_G)$ is a gyrolinear space.

Example 10. Define the binary operation $\oplus_M$ on $\mathbb{P}_1 = \mathbb{R}_+$ by

$$a \oplus_M b = \max\{a, b\} \quad \text{for all } a, b \in \mathbb{R}_+,$$

then $(\mathbb{P}_n, \oplus_M)$ is a uniquely 2-divisible commutative semi-group. Denote by $\mathrm{Max}(a,b)$ the semi-group midpoint of a and b, that is,

$$\mathrm{Max}(a,b) = \frac{1}{2} \otimes_M (a \oplus_M b) = \max\{a, b\},$$

then $\mathrm{Max}(\cdot,\cdot)$ is a mean on $\mathbb{R}_+$. Moreover, define the map $\otimes_H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\lambda \otimes_H a = a \quad \text{for all } a \in \mathbb{P}_n \text{ and } \lambda \in \mathbb{R}_+,$$

then $(\mathbb{P}_n, \oplus_H, \otimes_H)$ is a semi-linear space. In particular, $(\mathbb{P}_n, \oplus_M)$ is not left-cancellative or right-cancellative.

6 A theorem

Theorem 11. *Let $(\mathbb{P}_n, \oplus)$ be a uniquely 2-divisible commutative semi-group. Suppose that $M(a,b) = \frac{1}{2} \otimes (a \oplus b)$ is a mean on $\mathbb{P}_n$. Then the following* (i) *and* (ii) *are equivalent to each other.*

(i) $(\mathbb{P}_n, \oplus)$ is (left and right) cancellative.

(ii) $b \neq c$ implies $M(a,b) \neq M(a,c)$.

Corollary 12. *Let $(\mathbb{P}_n, \oplus)$ be a uniquely 2-divisible commutative group. If $M(a,b) = \frac{1}{2} \otimes (a \oplus b)$ is a mean on $\mathbb{P}_n$, then $b \neq c$ implies $M(a,b) \neq M(a,c)$.*

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Cauchy-Bunyakovsky-Schwarz type inequalities related to Möbius operations

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1 導入

（実または複素）内積空間 $(\mathbb{V}, \langle\cdot,\cdot\rangle)$ における Cauchy-Bunyakovsky-Schwarz の不等式 ([C], [B], [S]. 以下 CBS 不等式という）

$$|\langle u,v\rangle| \leq \langle u,u\rangle^{\frac{1}{2}}\langle v,v\rangle^{\frac{1}{2}} \qquad (u,\, v \in \mathbb{V})$$

等号が成立するのは u, v が線形従属のとき, そのときに限る

は数学における最も基本的な不等式のひとつである.

この報告では, 内積空間の 3 つのベクトルとひとつの正数パラメータに対して成り立つ不等式で, CBS 不等式のある種の拡張となっていて Möbius 演算に関係しているものについて述べる.

複素平面の単位開円板 $\mathbb{D} = \{a \in \mathbb{C};\ |a| < 1\}$ における Möbius の和は

$$a \oplus_{\mathrm{M}} b = \frac{a+b}{1+\overline{a}b} \qquad (a, b \in \mathbb{D})$$

であり, 数学の広く様々な分野に現れる. Möbius の和は以前から知られていたが, その群のような構造は, Einstein の特殊相対論の脈絡で Ungar によって 1988 年に明らかにされるまで, 気付かれていなかった. さらに Ungar は任意の実内積空間の開球に Möbius の和を拡張し, また Möbius のスカラー倍を導入して, ベクトル空間のような構造をもつ gyrovector space の概念を確立した.

手短に Möbius gyrovector space の定義を思い出そう. 我々の今回の結果は Möbius の演算や gyrovector space の理論を使わなくても述べることができるが, それらは我々の重要なモチベーションおよび背景であり, その記法は我々の不等式の記述を著しく簡単にする. 抽象的な (gyrocommutative) gyrogroup, gyrovector space の定義や基本的事項については, 例えば [U] を参照していただきたい.

Möbius Gyrovector Spaces.[U] $\mathbb{V}$ を任意の実内積空間, 固定された正の数 s に対して

$$\mathbb{V}_s = \{\boldsymbol{a} \in \mathbb{V};\ ||\boldsymbol{a}|| < s\}$$

とする. Möbius の和および Möbius のスカラー倍は

$$\boldsymbol{a} \oplus_{\mathrm{M}} \boldsymbol{b} = \frac{\left(1 + \frac{2}{s^2}\langle \boldsymbol{a}, \boldsymbol{b}\rangle + \frac{1}{s^2}||\boldsymbol{b}||^2\right)\boldsymbol{a} + \left(1 - \frac{1}{s^2}||\boldsymbol{a}||^2\right)\boldsymbol{b}}{1 + \frac{2}{s^2}\langle \boldsymbol{a}, \boldsymbol{b}\rangle + \frac{1}{s^4}||\boldsymbol{a}||^2||\boldsymbol{b}||^2}$$

$$r \otimes_{\mathrm{M}} \boldsymbol{a} = s\ \tanh\left(r \tanh^{-1}\frac{||\boldsymbol{a}||}{s}\right)\frac{\boldsymbol{a}}{||\boldsymbol{a}||} \quad (\text{if } \boldsymbol{a} \neq \boldsymbol{0}), \qquad r \otimes_{\mathrm{M}} \boldsymbol{0} = \boldsymbol{0}$$

for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}_s$, $r \in \mathbb{R}$ によって定義される.
公理 (VV) の, $||\mathbb{V}_s|| = (-s, s)$ における演算 $\oplus_{\mathrm{M}}$, $\otimes_{\mathrm{M}}$（同一の記号が使われる）は

$$a \oplus_{\mathrm{M}} b = \frac{a+b}{1 + \frac{1}{s^2}ab}$$

$$r \otimes_{\mathrm{M}} a = s \tanh\left(r \tanh^{-1}\frac{a}{s}\right)$$

for all $a, b \in (-s, s)$, $r \in \mathbb{R}$ によって定義される.
このとき, $(\mathbb{V}_s, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$ は gyrovector space となる. $\oplus_{\mathrm{M}}, \otimes_{\mathrm{M}}$ をそれぞれ単に $\oplus, \otimes$ と書く. パラメータ s を明示したい場合は $\oplus_s, \otimes_s$ と書く.

一般には, 演算は可換でも, 結合的でも, 分配的でもないことに注意する:

$$\boldsymbol{a} \oplus \boldsymbol{b} \neq \boldsymbol{b} \oplus \boldsymbol{a}$$

$$\boldsymbol{a} \oplus (\boldsymbol{b} \oplus \boldsymbol{c}) \neq (\boldsymbol{a} \oplus \boldsymbol{b}) \oplus \boldsymbol{c}$$

$$r \otimes (\boldsymbol{a} \oplus \boldsymbol{b}) \neq r \otimes \boldsymbol{a} \oplus r \otimes \boldsymbol{b}$$

$$t(\boldsymbol{a} \oplus \boldsymbol{b}) \neq t\boldsymbol{a} \oplus t\boldsymbol{b}.$$

しかし, 左（および右）ジャイロ結合法則, ジャイロ交換法則, スカラー分配法則, スカラー結合法則などがあるように, gyrovector space は解明すべき豊かな対称性を有している.

$s \to \infty$ とすると $\mathbb{V}_s$ は全空間 $\mathbb{V}$ に拡大し, 演算 $\oplus_s, \otimes_s$ は通常のベクトル和, スカラー倍に近づく.
Proposition.[U]

$$\boldsymbol{a} \oplus_s \boldsymbol{b} \to \boldsymbol{a} + \boldsymbol{b} \quad (s \to \infty)$$

$$r \otimes_s \boldsymbol{a} \to r\boldsymbol{a} \quad (s \to \infty).$$

Notation.[U] It is obvious that $-u$ is the inverse element of u with respect to $\oplus$ as well. As in group theory, we use the notation

$$\boldsymbol{a} \ominus \boldsymbol{b} = \boldsymbol{a} \oplus (-\boldsymbol{b}).$$

The Möbius gyrodistance function d on a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ is defined by the equation

$$d(\boldsymbol{a}, \boldsymbol{b}) = ||\boldsymbol{b} \ominus \boldsymbol{a}||.$$

Moreover, the Poincaré distance function h on the ball $\mathbb{V}_s$ is introduced by the equation

$$h(\boldsymbol{a}, \boldsymbol{b}) = \tanh^{-1} \frac{d(\boldsymbol{a}, \boldsymbol{b})}{s}.$$

Theorem.[U] The function h satisfies the triangle inequality, so that $(\mathbb{V}_s, h)$ is a metric space. It is also complete as a metric space provided $\mathbb{V}$ is complete.

Proposition. Let $s > 0$. The following formulae hold

(i) $\dfrac{\boldsymbol{a}}{s} \oplus_1 \dfrac{\boldsymbol{b}}{s} = \dfrac{\boldsymbol{a} \oplus_s \boldsymbol{b}}{s}$

(ii) $||\boldsymbol{a} \oplus_s \boldsymbol{b}||^2 = \dfrac{||\boldsymbol{a}||^2 + 2\langle \boldsymbol{a}, \boldsymbol{b}\rangle + ||\boldsymbol{b}||^2}{1 + \frac{2}{s^2}\langle \boldsymbol{a}, \boldsymbol{b}\rangle + \frac{1}{s^4}||\boldsymbol{a}||^2||\boldsymbol{b}||^2}$

for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}_s$.

Ungar の (real inner product) gyrovector space では交換法則, 結合法則, 分配法則がそのままでは成り立たない. しかし, 最近の研究によって, 或いは, 自然にそうなっているからというべきなのか, Möbius gyrovector space については Hilbert 空間との間に強いアナロジーがはたらく事が知られてきた. 閉部分空間に関する直交分解, 閉部分空間さらには閉凸集合の最近点, 正規直交基底による直交展開, 線形作用素などの counterpart が考察されている. これらについては近年の関数環報告集 [AW], [W1] を参照していただきたい.

2　Möbius の演算に関連した CBS 型不等式

Möbius の和に関連した CBS 型不等式として, 我々は次の定理を得ることができた.

Theorem.[W3] Let $\mathbb{V}$ be a complex inner product space and let $w \in \mathbb{V}$ be a fixed element with $||w|| \leq 1$. For any u, $v \in \mathbb{V}$ and for any $s > \max\{||u||, ||v||\}$, the following inequality holds

$$\left| \frac{\langle u, w\rangle - \langle v, w\rangle}{1 - \frac{1}{s^2}\overline{\langle u, w\rangle}\langle v, w\rangle} \right| \leq \sqrt{\frac{||u||^2 - 2\mathrm{Re}\langle u, v\rangle + ||v||^2}{1 - \frac{2}{s^2}\mathrm{Re}\langle u, v\rangle + \frac{1}{s^4}||u||^2||v||^2}}. \tag{1}$$

The equality holds if and only if one of the following conditions is satisfied :

(i) $u = v$

(ii) $||w|| = 1$ and $u = \lambda w, v = \mu w$ for some $\lambda, \mu \in \mathbb{C}$.

Remark. • 実内積空間でも同様である. その不等式は次のように述べられる.

$$|\langle u, w\rangle \ominus_s \langle v, w\rangle| \leq ||u \ominus_s v||$$

for any $||u||, ||v|| < s$, $||w|| \leq 1$.

• v, w を $0, \frac{w}{||w||}$ で置き換えると古典的な CBS 不等式を得る. また, $s \to \infty$ とすることにより極限として古典的な CBS 不等式が復元される.

- 不等式 (1) を次のように示すことはできない.（最初の不等号が一般には成り立たない.）

$$|\langle u, w\rangle \ominus_s \langle v, w\rangle| \le |\langle u \ominus_s v, w\rangle| \le ||u \ominus_s v||\,||w|| \le ||u \ominus_s v||\,.$$

Example. $\mathbb{C}$ において $\langle u, v\rangle = u\overline{v}$,

$$u = \frac{1}{\sqrt{2}}, \quad v = -\frac{1}{\sqrt{2}}, \quad w = \frac{1}{\sqrt{2}}$$

とすると

$$\left|\frac{\langle u, w\rangle - \langle v, w\rangle}{1 - \overline{\langle u, w\rangle}\langle v, w\rangle}\right| = \frac{4}{5} > \frac{2}{3} = \sqrt{\frac{||u||^2 - 2\mathrm{Re}\langle u, v\rangle + ||v||^2}{1 - 2\mathrm{Re}\langle u, v\rangle + ||u||^2||v||^2}}\,||w||.$$

このように, $||u||, ||v|| < 1$, $||w|| < 1$ に対して不等式

$$\left|\frac{\langle u, w\rangle - \langle v, w\rangle}{1 - \overline{\langle u, w\rangle}\langle v, w\rangle}\right| \le \sqrt{\frac{||u||^2 - 2\mathrm{Re}\langle u, v\rangle + ||v||^2}{1 - 2\mathrm{Re}\langle u, v\rangle + ||u||^2||v||^2}}\,||w||$$

は一般に成立しない. 筆者が 2018 年 6 月の米沢数学セミナーでこれらを発表したとき, 高橋眞映先生は次の質問をなされた.

Qusetion.(S.-E. Takahasi) Is there any constant $C > 1$ s.t.

$$\left|\frac{\langle u, w\rangle - \langle v, w\rangle}{1 - \overline{\langle u, w\rangle}\langle v, w\rangle}\right| \le C\sqrt{\frac{||u||^2 - 2\mathrm{Re}\langle u, v\rangle + ||v||^2}{1 - 2\mathrm{Re}\langle u, v\rangle + ||u||^2||v||^2}}\,||w||$$

for any $||u||, ||v|| < 1$, $||w|| \le 1$?

これに答えようとして次の定理が得られた.

Theorem.[W4] Let $\mathbb{V}$ be a complex inner product space. For any $u, v \in \mathbb{V}$, $s > \max\{||u||, ||v||\}$ and $w \in \mathbb{V}$ with $||w|| \le 1$, the following inequality holds

$$\left|\frac{\langle u, w\rangle - \langle v, w\rangle}{1 - \frac{1}{s^2}\overline{\langle u, w\rangle}\langle v, w\rangle}\right| \le \sqrt{\frac{||u||^2 - 2\mathrm{Re}\langle u, v\rangle + ||v||^2}{1 - \frac{2}{s^2}\mathrm{Re}\langle u, v\rangle + \frac{1}{s^4}||u||^2||v||^2}} \cdot \frac{2||w||}{1 + ||w||^2}. \tag{2}$$

The equality holds if and only if one of the following conditions is satisfied :

(i) $u = v$

(ii) $w = 0$

(iii) $||w|| = 1$ and $u = \lambda w, v = \mu w$ for some $\lambda, \mu \in \mathbb{C}$.

$\dfrac{2||w||}{1+||w||^2} \leq 2||w||$ だから次が成り立ち, Takahasi の問いに $C=2$ として肯定的な回答を与える.

Corollary. If $||u||, ||v|| < 1$, $||w|| \leq 1$, then

$$\left|\frac{\langle u,w\rangle - \langle v,w\rangle}{1-\overline{\langle u,w\rangle}\langle v,w\rangle}\right| \leq 2\sqrt{\frac{||u||^2 - 2\mathrm{Re}\langle u,v\rangle + ||v||^2}{1-2\mathrm{Re}\langle u,v\rangle + ||u||^2||v||^2}}\,||w||.$$

Remark. • 不等式 (2) は不等式 (1) の改良である. $\dfrac{2||w||}{1+||w||^2} \leq 1$ だから.
• 実内積空間でも同様である.

次の命題は, 上記の系の右辺の定数 2 はある意味で最良であることを示している.

Proposition. For any constant $C < 2$, there exist elements $u, v, w \in \mathbb{V}$ satisfying $||u||, ||v|| < 1$, $||w|| \leq 1$ and

$$\left|\frac{\langle u,w\rangle - \langle v,w\rangle}{1-\overline{\langle u,w\rangle}\langle v,w\rangle}\right| > C\sqrt{\frac{||u||^2 - 2\mathrm{Re}\langle u,v\rangle + ||v||^2}{1-2\mathrm{Re}\langle u,v\rangle + ||u||^2||v||^2}}||w||.$$

次の命題は, 不等式 (2) を古典的な CBS 不等式とそれ以外の部分に単純に分解して証明できないことを意味している.

Proposition. For any constant $C > 0$, there exist elements $u, v, w \in \mathbb{V}$ satisfying $||u||, ||v|| < 1$, $||w|| \leq 1$ and

$$\left|\frac{1}{1-\overline{\langle u,w\rangle}\langle v,w\rangle}\right| > C\sqrt{\frac{1}{1-2\mathrm{Re}\langle u,v\rangle + ||u||^2||v||^2}} \cdot \frac{2}{1+||w||^2}.$$

Möbius 和および Möbius スカラー倍双方に関連したある離散的 Cauchy 型不等式が [W2] で得られている. 次の定理は, 内積空間と Möbius 和および Möbius スカラー倍双方との関係の脈絡において, CBS 型不等式の最も自然な拡張とみなされ得る.

Theorem.[W4] Let $\mathbb{V}$ be a complex inner product space. For any u, $v \in \mathbb{V}$, $s > \max\{||u||, ||v||\}$ and $w \in \mathbb{V}$ with $||w|| \leq 1$, the following inequality holds

$$\left|\frac{\langle u,w\rangle - \langle v,w\rangle}{1-\frac{1}{s^2}\overline{\langle u,w\rangle}\langle v,w\rangle}\right| \leq ||w|| \otimes_s \sqrt{\frac{||u||^2 - 2\mathrm{Re}\langle u,v\rangle + ||v||^2}{1-\frac{2}{s^2}\mathrm{Re}\langle u,v\rangle + \frac{1}{s^4}||u||^2||v||^2}}. \tag{3}$$

The equality holds if and only if one of the following conditions is satisfied :

(i) $u = v$

(ii) $w = 0$

(iii) $||w|| = 1$ and $u = \lambda w, v = \mu w$ for some $\lambda, \mu \in \mathbb{C}$.

Remark. • 不等式 (3) は不等式 (1) の改良である. $0 \le r \le 1, 0 \le a < s$ ならば $r \otimes_s a \le a$ だから.

• 実内積空間でも同様である. その不等式は次のように述べられる.

Let $s > 0$. For any elements $u, v, w \in \mathbb{V}$ with $||u||, ||v|| < s$, $||w|| \le 1$,

$$|\langle u, w\rangle \ominus_s \langle v, w\rangle| \le ||w|| \otimes_s ||u \ominus_s v|| . \tag{4}$$

In other words,

$$\tanh^{-1} \frac{|\langle u, w\rangle \ominus_s \langle v, w\rangle|}{s} \le ||w|| \tanh^{-1} \frac{||u \ominus_s v||}{s}$$

or

$$h\left(\langle u, w\rangle, \langle v, w\rangle\right) \le h\left(u, v\right) ||w||.$$

• 不等式 (3) や (4) で $s \to \infty$ とすると古典的な CBS 不等式

$$|\langle u, w\rangle - \langle v, w\rangle| \le ||w|| ||u - v||$$

が復元される.

また, 次が成り立つ.

Theorem. Let $\mathbb{V}$ be a complex inner product space and let $w \in \mathbb{V}$ be an arbitrary fixed element with $||w|| \le 1$. If K is a constant satisfying

$$\left|\frac{\langle u, w\rangle - \langle v, w\rangle}{1 - \overline{\langle u, w\rangle}\langle v, w\rangle}\right| \le K \otimes_1 \sqrt{\frac{||u||^2 - 2\mathrm{Re}\langle u, v\rangle + ||v||^2}{1 - 2\mathrm{Re}\langle u, v\rangle + ||u||^2||v||^2}}.$$

for any any element $u, v \in \mathbb{V}$ with $||u||, ||v|| < 1$, then $||w|| \le K$.

最後に, 不等式 (4) の応用として Riesz の表現定理のひとつの counterpart を提示する.

Definition. Let $\mathbb{V}$ be an inner product space. For any map $f : \mathbb{V}_1 \to (-1, 1)$, define $f_s : \mathbb{V}_s \to (-s, s)$ by

$$f_s(\boldsymbol{x}) = sf\left(\frac{\boldsymbol{x}}{s}\right)$$

for any element $\boldsymbol{x} \in \mathbb{V}_s$.

Theorem.[W5] Let $\mathbb{V}$ be a real inner product space, $\boldsymbol{c} \in \mathbb{V}$ with $||\boldsymbol{c}|| \le 1$, and consider the functional $f : \mathbb{V}_1 \to (-1, 1)$ defined by

$$f(\boldsymbol{x}) = \langle \boldsymbol{x}, \boldsymbol{c}\rangle$$

for any element $\boldsymbol{x} \in \mathbb{V}_1$. Then,

(i) For any $\epsilon > 0$, f_s satisfies the following conditions:

$$-\{f_s(\boldsymbol{x}) \oplus_s f_s(\boldsymbol{y})\} \oplus_s f_s(\boldsymbol{x} \oplus_s \boldsymbol{y}) = o(s^{-2+\epsilon}) \quad (s \to \infty)$$
$$-\{r \otimes_s f_s(\boldsymbol{x})\} \oplus_s f_s(r \otimes_s \boldsymbol{x}) = o(s^{-2+\epsilon}) \quad (s \to \infty)$$

for any element $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}$ and any real number $r \in \mathbb{R}$. Here, $f(s) = o(s^{\alpha})$ $(s \to \infty)$ means that $\dfrac{f(s)}{s^{\alpha}} \to 0$ $(s \to \infty)$.

(ii) The following formula

$$\sup_{\boldsymbol{x},\boldsymbol{y} \in \mathbb{V}_1,\, \boldsymbol{x} \neq \boldsymbol{y}} \frac{h(f(\boldsymbol{x}), f(\boldsymbol{y}))}{h(\boldsymbol{x}, \boldsymbol{y})} = ||\boldsymbol{c}||$$

holds.

Theorem.[W5] Let $\mathbb{V}$ be a real Hilbert space. Suppose that $f : \mathbb{V}_1 \to (-1, 1)$ satisfies the following conditions

$$-\{f_s(\boldsymbol{x}) \oplus_s f_s(\boldsymbol{y})\} \oplus_s f_s(\boldsymbol{x} \oplus_s \boldsymbol{y}) \to 0 \quad (s \to \infty)$$
$$-\{r \otimes_s f_s(\boldsymbol{x})\} \oplus_s f_s(r \otimes_s \boldsymbol{x}) \to 0 \quad (s \to \infty)$$

and

$$0 \leq \sup_{\boldsymbol{x},\boldsymbol{y} \in \mathbb{V}_1,\, \boldsymbol{x} \neq \boldsymbol{y}} \frac{h(f(\boldsymbol{x}), f(\boldsymbol{y}))}{h(\boldsymbol{x}, \boldsymbol{y})} \leq 1.$$

Then,

(i) For any $\boldsymbol{x} \in \mathbb{V}$, $\lim_{s \to \infty} f_s(\boldsymbol{x})$ exists as a real number.

(ii) There exists a unique element $\boldsymbol{c} \in \mathbb{V}$ satisfying

$$\lim_{s \to \infty} f_s(\boldsymbol{x}) = \langle \boldsymbol{x}, \boldsymbol{c} \rangle \quad (\boldsymbol{x} \in \mathbb{V}) \qquad \text{and} \qquad \sup_{\boldsymbol{x},\boldsymbol{y} \in \mathbb{V}_1,\, \boldsymbol{x} \neq \boldsymbol{y}} \frac{h(f(\boldsymbol{x}), f(\boldsymbol{y}))}{h(\boldsymbol{x}, \boldsymbol{y})} = ||\boldsymbol{c}||.$$

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Estimates for the weighted polyharmonic Bergman kernel and their application(announcement)

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Abstract

We consider the Bergman type space with respect to polyharmonic functions. The purpose of this article is the announcement of the author's paper [6]. Based on [6], this article describes the estimates for the reproducing kernel of the polyharmonic Bergman kernel, the Gleason problem and the Lipschitz type characterization for polyharmonic Bergman space without proofs and details.

Throughout this article, let $\mathbb{B}$ be the open unit ball in the Euclidean space $\mathbb{R}^N$. For $m \in \mathbb{N}$, $1 \leq p < \infty$ and $\alpha > -1$, the weighted m-polyharmonic Bergman space $b_\alpha^{m,p}(\mathbb{B})$ is the set of polyharmonic function f of degree m in $\mathbb{B}$ such that

$$\|f\|_{b_\alpha^{m,p}} := \left(\int_{\mathbb{B}} |f(x)|^p (1-|x|^2)^\alpha dx \right)^{1/p} < \infty.$$

In particular, when $p = 2$, the weighted m-polyharmonic Bergman space $b_\alpha^{m,2}(\mathbb{B})$ is a reproducing kernel Hilbert space. We define the weighted true m-polyharmonic Bergman space $b_\alpha^{(m),2}(\mathbb{B})$ by

$$b_\alpha^{(m),2}(\mathbb{B}) = b_\alpha^{m,2}(\mathbb{B}) \ominus b_\alpha^{m-1,2}(\mathbb{B}).$$

We denote the reproducing kernels of $b_\alpha^{m,2}(\mathbb{B})$ and $b_\alpha^{(m),2}(\mathbb{B})$ by $R_{m,\alpha}(x,y)$ and $R_{(m),\alpha}(x,y)$, respectively. We call $R_{m,\alpha}(x,y)$ the weighted m-polyharmonic Bergman kernel. For simplicity, when $\alpha = 0$, we omit to write α, for example, $b^{m,p}(\mathbb{B}) := b_0^{m,p}(\mathbb{B})$.

On the theory of Bergman type space, the estimates for the reproducing kernel play important roles, for examples[1, 3]. Hence, we should calculate the estimates for $R_{m,\alpha}(x,y)$ and $R_{(m),\alpha}(x,y)$. For $m = 2$, T.[5] gave the estimates and explicit form for the biharmonic Bergman kernel $R_{2,\alpha}(x,y)$. In [6], we give the estimates for $R_{m,\alpha}(x,y)$ based on Pavlović's results[4].

Theorem 1 (Theorem 1.2 in [6]) *For $m \in \mathbb{N}$ and $\alpha > -1$, there exists a positive constant C such that*

$$|R_{m,\alpha}(x,y)| \leq \frac{C}{[x,y]^{\frac{N+\alpha}{2}}} \quad \text{and} \quad |\nabla_x R_{m,\alpha}(x,y)| \leq \frac{C}{[x,y]^{\frac{N+\alpha+1}{2}}}$$

for $x, y \in \mathbb{B}$, where $[x,y] = 1 - 2x\cdot y + |x|^2|y|^2$ and $x \cdot y = \sum_{i=1}^N x_i y_i$ for $x = (x_1, x_2, \cdots, x_N)$ and $y = (y_1, y_2, \cdots, y_N)$.

As an application of Theorem 1, we obtain the estimates for derivative of unweighted m-polyharmonic Bergman functions.

Lemma 1 (Lemma 4.2 in [6]) *Assume* $1 \leq p < \infty$. *One has*

$$\|f - f(0)\|_{b^{m,p}} \approx \|(1 - |x|^2)|\nabla f|\|_{L^p}$$

for $f \in b^{m,p}(\mathbb{B})$.

By Lemma 1, we mention the Gleason problem and the Lipschitz type characterization for polyharmonic Bergman space.

Theorem 2 (Gleason problem, Theorem 4.1 in [6]) *For* $1 \leq p < \infty$ *and* $f \in b^{m,p}(\mathbb{B})$, *there exist functions* $g_j \in b^{m,p}(\mathbb{B})$ $(j = 1, \cdots, N)$ *such that*

$$f(x) - f(0) = \sum_{j=1}^{N} x_j g_j(x).$$

Theorem 3 (Lipschitz type characterization, Theorem 4.2 in [6]) *Let* $1 \leq p < \infty$ *and* $f \in H^m(\mathbb{B})$. *Then,* f *belongs to* $b^{m,p}(\mathbb{B})$ *if and only if there exists a function* $g \in L^p(\mathbb{B}, (1-|x|^2)^p dx)$ *such that*

$$|f(x) - f(y)| \leq |x - y| \, (g(x) + g(y))$$

for any $x, y \in \mathbb{B}$.

Remark 1. After the conference, the author knew a paper [2]. In [2], Lemma 1 is shown without estimates for the reproducing kernel. The author thanks Professor M. Pavlović for introducing a paper [2].

Remark 2. At the conference, the author could not calculate the lower estimate for $R_{(m),\alpha}(x, x)$. After that, we obtain the lower estimates for the unweighted kernel:

$$R_{(m)}(x, x) \geq \frac{C}{(1 - |x|^2)^N}$$

for some constant C. If we make further progress about the lower estimate for $R_{(m),\alpha}(x, x)$, the author would like to talk it at next Conference on Function Algebras.

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Mean Lipschitz conditions and growth of area integral means of functions in Bergman spaces[1]

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1 Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and dA the normalized area measure on $\mathbb{D}$. Let $H(\mathbb{D})$ be the set of all analytic functions on $\mathbb{D}$. The classical Hardy space is denoted by H^p and the Bergman space is denoted by A^p ($p \in (0, \infty)$). It is well known that the function in H^p has many properties. One of properties of $f \in H^p$ is the Hardy and Littlewood theorem related to boundary value functions. For $0 < p \leq \infty$, $0 \leq r < 1$ and $f \in H(\mathbb{D})$, the integral mean $M_p(r, f)$ is defined by

$$M_p(r, f) = \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}$$

and

$$M_\infty(r, f) = \sup_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|.$$

Let f^* denote the radial limit of $f \in H^p$ and put $\tau_t(f^*)(\theta) = f^*(\theta + t)$ for $t \in \mathbb{R}$. Hardy and Littlewood proved the following theorem.

Theorem A. *Let $1 \leq p < \infty$, $0 < \alpha \leq 1$. For $f \in H^p$ the following conditions are equivalent:*

(a) $\|\tau_t(f^*) - f^*\|_{H^p} = O(|t|^\alpha)$ *as* $t \to 0$,

(b) $M_p(r, f') = O((1 - r)^{\alpha - 1})$ *as* $r \to 1^-$.

The above condition (a) is called the mean Lipschitz condition, which is appeared in the definition of the analytic Lipschitz space. For $f \in H(\mathbb{D})$ and $0 \leq r < 1$, consider the dilated functions f_r of f, that is $f_r(z) = f(rz)$ ($z \in$

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$\mathbb{D}$). It is also known that $\|f_r - f\|_{H^p} \to 0$ as $r \to 1^-$ if $f \in H^p$. According to [2], Storozhenko [5] proved that the mean Lipschitz condition (a) of Theorem A is also equivalent to $\|f_r - f\|_{H^p} = O((1-r)^\alpha)$ as $r \to 1^-$. Hence we can collect their results as follows.

Theorem B. *Let $1 \leq p < \infty$, $0 < \alpha \leq 1$. For $f \in H^p$ the following conditions are equivalent:*

(a) $\|\tau_t(f^*) - f^*\|_{H^p} = O(|t|^\alpha)$ *as* $t \to 0$,

(b) $M_p(r, f') = O((1-r)^{\alpha-1})$ *as* $r \to 1^-$,

(c) $\|f_r - f\|_{H^p} = O((1-r)^\alpha)$ *as* $r \to 1^-$.

P. Galanopoulos et al. [2] have recently proved that the function in the classical Bergman space A^p $(1 \leq p < \infty)$ has the same property as Theorem B. To adapt mean Lipschitz condition and integral mean over the unit circle for $f \in A^p$, they introduced the following rotation function and area integral mean of f:

$$r_t(f)(z) = f(e^{it}z) \quad (t \in \mathbb{R})$$

and

$$A_p(r, f) = \|f_r\|_{A^p} = \left(\int_{\mathbb{D}} |f(rz)|^p dA(z) \right)^{1/p} \qquad (r \in [0,1)).$$

For $f \in H(\mathbb{D})$, if there exists f^* a.e. on $\partial\mathbb{D}$, then $\tau_t(f^*) = (r_t(f))^*$. Thus this notation r_t can translate the condition (a) of Theorem B into the version of Bergman space. They proved that the same result as Theorem A also holds for $f \in A^p$ $(1 \leq p < \infty)$.

Theorem C.([2]) *Let $1 \leq p < \infty$, $0 < \alpha \leq 1$. For $f \in A^p$ the following conditions are equivalent:*

(a) $\|r_t(f) - f\|_{A^p} = O(|t|^\alpha)$ *as* $t \to 0$,

(b) $A_p(r, f') = O((1-r)^{\alpha-1})$ *as* $r \to 1^-$,

(c) $\|f_r - f\|_{A^p} = O((1-r)^\alpha)$ *as* $r \to 1^-$.

In [2], they also mentioned that the analogue of Theorem C is valid in the Dirichlet space and the disk algebra.

2 Results

Motivated by their study, we will consider the same problem for weighted Bergman spaces and related spaces. For a given positive continuous function σ on $[0,1)$, we extend it by $\sigma(z) = \sigma(|z|)$ for $z \in \mathbb{D}$. We call such σ a weight function on $\mathbb{D}$. For a weight function σ, the weighted Bergman space $A^p_\sigma(\mathbb{D})$ $(p \in (0,\infty))$ is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{p,\sigma} = \left(\int_{\mathbb{D}} |f(z)|^p \sigma(z) dA(z) \right)^{1/p} < \infty.$$

For the case $p = \infty$, we will introduce the related space $A^\infty_\sigma(\mathbb{D})$ as follows:

$$A^\infty_\sigma(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) \, : \, \|f\|_{\infty,\sigma} = \sup_{z\in\mathbb{D}} |f(z)|\sigma(z) < \infty \right\}.$$

If we assume some conditions on weight σ, then we find that $f \in A^p_\sigma(\mathbb{D})$ $(0 < p < \infty)$ has the property $\|f_r - f\|_{p,\sigma} \to 0$ as $r \to 1^-$. For $p = \infty$, we consider the subspace of $A^\infty_\sigma(\mathbb{D})$ such that a function f satisfies a vanishing property $f(z)\sigma(z) \to 0$ as $|z| \to 1^-$. Then such a function f also has the property $\|f_r - f\|_{\infty,\sigma} \to 0$ as $r \to 1^-$. Since a function in H^p or A^p has the same the limiting behavior, it is expected that $f \in A^p_\sigma(\mathbb{D})$ also has the same properties as Theorem C. In the spirit of the result in [2], we shall define the weighted area integral mean $A^\sigma_p(r, f)$ for $f \in H(\mathbb{D})$ and $0 \le r < 1$. We put

$$A^\sigma_p(r, f) = \|f_r\|_{p,\sigma} = \begin{cases} \left(\displaystyle\int_{\mathbb{D}} |f(rz)|^p \sigma(z) dA(z) \right)^{1/p} & \text{if } 0 < p < \infty, \\ \displaystyle\sup_{z\in\mathbb{D}} |f(rz)|\sigma(z) & \text{if } p = \infty. \end{cases}$$

By a simple calculation, we find that

$$\left\{A^\sigma_p(r, f)\right\}^p = 2\int_0^1 t\sigma(t)M^p_p(rt, f)dt = \frac{2}{r^2}\int_0^r s\sigma\left(\tfrac{s}{r}\right) M^p_p(s, f)ds$$

and

$$A^\sigma_\infty(r, f) = \sup_{0\le t<1} M_\infty(rt, f)\sigma(t) = \sup_{0\le s<r} M_\infty(s, f)\sigma\left(\tfrac{s}{r}\right).$$

Now we introduce the notion of an admissible weight function. The following definition is due to Kellay and Lefèvre [3] essentially. A weight function σ is called admissible if σ satisfies

(W_1) σ is non-increasing,

(W_2) $\sigma(r)/(1 - r^2)^{1+\delta}$ is non-decreasing for some $\delta > 0$,

(W_3) $\sigma(r) \to 0$ as $r \to 1^-$.

The typical example of admissible weight is $\sigma(r) = (1 - r^2)^\alpha$ $(\alpha > 0)$.

Next we introduce the Békollé weight which is an analogue of the Muckenhoupt weight. We quote the following notion from Luecking's paper [4]. For each $\alpha > -1$, let dA_α denote the normalized measure on $\mathbb{D}$ defined by $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$. For $p > 1$ and $\alpha > -1$, the class $B_p(\alpha)$ consists of all weight functions σ with the property that there is a positive constant C such that for every $a \in \mathbb{D}$,

$$\left(\int_{S(a)} \sigma dA_\alpha\right) \cdot \left(\int_{S(a)} \sigma^{-\frac{p'}{p}} dA_\alpha\right)^{\frac{p}{p'}} \le C \left\{A_\alpha(S(a))\right\}^p,$$

where $1/p + 1/p' = 1$, and $S(a) = \{\varphi_a(z) \, : \, \mathrm{Re}(z\overline{a}) \le 0\}$. Note that we put $S(0) = \mathbb{D}$. Aleman and Constantin [1, Theorem 3.1] proved that if a weight

σ satisfies $\sigma(z)/(1-|z|^2)^\alpha \in B_{p_0}(\alpha)$ for some $p_0 > 1$ and $\alpha > -1$, then the norm $\|f\|_{p,\sigma}^p$ is equivalent to

$$|f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p \sigma(z) dA(z).$$

To estimate the growth of f' for $f \in A_\sigma^p(\mathbb{D})$, we will need the above result. Hence we have to consider the condition:

(W_4) $\quad \frac{\sigma(z)}{(1-|z|^2)^\eta} \in B_{p_0}(\eta)$ for some $p_0 > 1$ and $\eta > -1$.

If an admissible weight function satisfies (W_4), we call it an admissible Békollé weight function.

When $1 \le p < \infty$, we will prove that the analogue of Theorem C is true for a function $f \in A_\sigma^p(\mathbb{D})$ with admissible Békollé weight. For the case $p = \infty$ we do not need the condition (W_4) in the argument for the space $A_\sigma^\infty(\mathbb{D})$. Namely the analogue of Theorem C holds for $f \in A_\sigma^\infty(\mathbb{D})$ with admissible weight.

Theorem 1 *Let σ be an admissible Békollé weight function, $1 \le p < \infty$, $0 < \alpha \le 1$ and $f \in A_\sigma^p(\mathbb{D})$. Then the following conditions are equivalent:*

(a) $\quad \|r_t(f) - f\|_{p,\sigma} = O(|t|^\alpha)$ *as* $t \to 0$,

(b) $\quad A_p^\sigma(r, f') = O((1-r)^{\alpha-1})$ *as* $r \to 1^-$,

(c) $\quad \|f_r - f\|_{p,\sigma} = O((1-r)^\alpha)$ *as* $r \to 1^-$.

Theorem 2 *Let σ be an admissible weight function, $0 < \alpha \le 1$ and $f \in A_\sigma^\infty(\mathbb{D})$. Then the following conditions are equivalent:*

(a) $\quad \|r_t(f) - f\|_{\infty,\sigma} = O(|t|^\alpha)$ *as* $t \to 0$,

(b) $\quad A_\infty^\sigma(r, f') = O((1-r)^{\alpha-1})$ *as* $r \to 1^-$,

(c) $\quad \|f_r - f\|_{\infty,\sigma} = O((1-r)^\alpha)$ *as* $r \to 1^-$.

Furthermore, we also consider the Bloch-type and the Zygmund-type space. By following in the normal weighted Bloch or Zygmund-type spaces, we will introduce the Bloch-type space $\mathcal{B}_\sigma(\mathbb{D})$ and the Zygmund-type space $\mathcal{Z}_\sigma(\mathbb{D})$ for an admissible weight function σ as follows:

$$\mathcal{B}_\sigma(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) \,:\, \sup_{z\in\mathbb{D}} |f'(z)|\sigma(z) < \infty \right\}$$

and

$$\mathcal{Z}_\sigma(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) \,:\, \sup_{z\in\mathbb{D}} |f''(z)|\sigma(z) < \infty \right\}.$$

Moreover, for $f \in \mathcal{B}_\sigma(\mathbb{D})$, its norm $\|f\|_{\mathcal{B}_\sigma}$ is defined by

$$\|f\|_{\mathcal{B}_\sigma} = |f(0)| + \|f'\|_{\infty,\sigma}.$$

Also the norm of $f \in \mathcal{Z}_\sigma(\mathbb{D})$ is defined by

$$\|f\|_{\mathcal{Z}_\sigma} = |f(0)| + |f'(0)| + \|f''\|_{\infty,\sigma}.$$

Since $f \in \mathcal{B}_\sigma(\mathbb{D})$ (or $\mathcal{Z}_\sigma(\mathbb{D})$) if and only if $f' \in A_\sigma^\infty(\mathbb{D})$ (or $f'' \in A_\sigma^\infty(\mathbb{D})$), it is expected that the same type result of Theorem 2 holds for these spaces. Instead of $A_\infty^\sigma(r, f')$ of (b) in Theorem 2, we consider the quantity

$$B^\sigma(r, F) = \|F_r\|_{\mathcal{B}_\sigma} = |F(0)| + r \sup_{z\in\mathbb{D}} |F'(rz)|\sigma(z) \tag{1}$$

for $F \in H(\mathbb{D})$ and $r \in (0, 1)$. Then the analogue of Theorem 2 is valid in the Bloch-type space.

Corollary 1 *Let σ be an admissible weight function, $0 < \alpha \leq 1$ and $f \in \mathcal{B}_\sigma(\mathbb{D})$. Then the following conditions are equivalent:*

(a) $\quad \|r_t(f) - f\|_{\mathcal{B}_\sigma} = O(|t|^\alpha)$ *as* $t \to 0$,

(b) $\quad B^\sigma(r, f') = O((1-r)^{\alpha-1})$ *as* $r \to 1^-$,

(c) $\quad \|f_r - f\|_{\mathcal{B}_\sigma} = O((1-r)^\alpha)$ *as* $r \to 1^-$.

In order to consider the Zygmund-type space, we also introduce $Z^\sigma(r, F) = \|F_r\|_{\mathcal{Z}_\sigma}$. Then we have that

$$Z^\sigma(r, f') = |f'(0)| + r|f''(0)| + r^2 A_\infty^\sigma(r, f'''). \tag{2}$$

By applying Theorem 2 to f'' $(\in A_\sigma^\infty(\mathbb{D}))$, we also obtain the following result.

Corollary 2 *Let σ be an admissible weight function, $0 < \alpha \leq 1$ and $f \in \mathcal{Z}_\sigma(\mathbb{D})$. Then the following conditions are equivalent:*

(a) $\quad \|r_t(f) - f\|_{\mathcal{Z}_\sigma} = O(|t|^\alpha)$ *as* $t \to 0$,

(b) $\quad Z^\sigma(r, f') = O((1-r)^{\alpha-1})$ *as* $r \to 1^-$,

(c) $\quad \|f_r - f\|_{\mathcal{Z}_\sigma} = O((1-r)^\alpha)$ *as* $r \to 1^-$.

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SCHUR PARAMETERS AND THE SPACE OF FINITE BLASCHKE PRODUCTS

TOSHIYUKI SUGAWA

Abstract. This is a preliminary version of the author's forthcoming paper. Our main result states that the Schur class with the topology of uniform convergence on compact subsets is homeomorphic to the closed unit ball of the space ℓ^2 with weak-$*$ topology. As an application, we show that the space of finite Blaschke products of degree d is homeomorphic to the $2d+1$ dimensional sphere.

1. Introduction

The function

$$T_a(z) = \frac{z-a}{1-\bar{a}z}$$

is an analytic automorphism of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (often called a disk automorphism) for every $a \in \mathbb{D}$. A function of the form

$$f(z) = e^{i\theta} \prod_{j=1}^{d} T_{a_j}(z)$$

with $\theta \in \mathbb{R}, a_j \in \mathbb{D}$ is called a (finite) Blaschke product of degree d. The following topological characterization is sometimes useful.

Lemma 1.1. *An analytic map $f : \mathbb{D} \to \mathbb{D}$ is a Blaschke product of degree d if and only if $f : \mathbb{D} \to \mathbb{D}$ is proper and of degree d.*

Here, a proper continuous mapping $f : D \to \Omega$ is said to be of degree d if the equation $f(z) = w$ has d roots in D for each $w \in \Omega$, counted according to multiplicity. (It is known, more strongly, that a holomorphic map $f : \mathbb{D} \to \mathbb{D}$ is a Blaschke product of order d if f is of degree d in the above sense.) In particular, we have the following corollary.

Corollary 1.2. *Let f be a Blaschke product of degree d. Then so is $L \circ f \circ M$ for disk automorphisms L and M.*

We denote by $\mathscr{B}_d$ the set of Blaschke products of degree d ($d = 0, 1, 2, \dots$). We set

$$\tilde{\mathscr{B}}_d = \bigcup_{j=0}^{d} \mathscr{B}_j.$$

The sets $\mathscr{B}_d$ and $\tilde{\mathscr{B}}_d$ will be equipped with the topology of locally uniform convergence on $\mathbb{D}$. Certainly it should be known that $\tilde{\mathscr{B}}_d$ is a compact Hausdorff (indeed, metrizable)

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topological space. However, it seems that the topological structure of $\tilde{\mathscr{B}}_d$ is not well studied. One difficulty is a degeneracy phenomenon. For instance, consider the disk automorphism

$$f_a(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}.$$

for a fixed $\theta \in \mathbb{R}$. When $a \to e^{it} \in \mathbb{T} = \partial\mathbb{D}$, the function f_a tends to the constant function $-e^{i(\theta+t)}$ locally uniformly on $\mathbb{D}$.

Obviously, $\mathscr{B}_0 \cong \partial\mathbb{D} = \mathbb{T} \cong \mathbb{S}^1$. Since an element f of $\mathscr{B}_1 = \mathrm{Aut}(\mathbb{D})$ has a representation of the form

$$f(z) = e^{i\theta}T_a(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z},$$

we see that $\mathscr{B}_1 \cong \mathbb{T} \times \mathbb{D}$, which is homeomorphic to a solid torus. Thus we expect $\tilde{\mathscr{B}}_1 = \mathscr{B}_0 \cup \mathscr{B}_1 \cong \mathbb{S}^3$. One of our main results is the following.

Theorem 1.3. *The space $\tilde{\mathscr{B}}_d$ is homeomorphic to the $(2d+1)$-dimensional sphere $\mathbb{S}^{2d+1}$.*

In what follows, we will show a more general results, from which the theorem will be deduced.

2. Schur parameters

The set

$$\mathscr{S} = \{f : \mathbb{D} \to \mathbb{C} \text{ holomorphic}, |f| \leq 1\}$$

is called the Schur class. Each function f in $\mathscr{S}$ can be expanded in the power series

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots = \sum_{n=0}^{\infty} c_n z^n.$$

However, the set of those coefficients $\{c_n\}$ is not convenient to parametrize the class $\mathscr{S}$. As we will see below, the Schur parameters are conveniet to describe the class $\mathscr{S}$.

For a function $f \in \mathscr{S} \setminus \mathscr{B}_0$, consider the new function

$$(\sigma f)(z) = \frac{1}{z} \cdot \frac{f(z)-\gamma}{1-\bar{\gamma}f(z)},$$

where

$$\gamma = f(0).$$

Since the origin is a removable singularity of $\sigma f(z)$, the function σf belongs to $\mathscr{S}$. When $f \in \mathscr{B}_0$, we set $\sigma f = 0$ as a convention. In this way, we define a mapping $\sigma : \mathscr{S} \to \mathscr{S}$. Observe that $\sigma(\mathscr{B}_d) = \mathscr{B}_{d-1}$ and $\sigma^{-1}(\mathscr{B}_{d-1}) = \mathscr{B}_d$ for $d \geq 1$. We define inductively f_n by $f_n = \sigma(f_{n-1})$ with $f_0 = f$. Let $\gamma_n = f_n(0) \in \overline{\mathbb{D}}$ $(n \geq 0)$. These are called the *Schur parameters* of $f \in \mathscr{S}$. In this paper,

$$\overrightarrow{\gamma} = \overrightarrow{\gamma}(f) = (\gamma_0, \gamma_1, \cdots) \in \overline{\mathbb{D}}^{\mathbb{N}_0}$$

will be called the *Schur vector* of f, where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Note that the Schur vector of the function $f_1 = \sigma f$ is $(\gamma_1, \gamma_2, \dots)$, which is the backward shift of the one-sided (unilateral) sequence $\overrightarrow{\gamma} = (\gamma_0, \gamma_1, \cdots)$. The following result due to Schur [2] is fundamental in our discussion. A comprehensive account on the Schur agorithm is found in the huge monograph [3].

Theorem 2.1 (Schur's theorem (1917))**.** *For a function $f \in \mathscr{S}$, the Schur vector $\overrightarrow{\gamma} = (\gamma_0, \gamma_1, \dots)$ satisfies one of the following two conditions:*

- $|\gamma_n| < 1$ *for all* n.
- $|\gamma_0| < 1, \dots, |\gamma_{n-1}| < 1, |\gamma_n| = 1, \gamma_{n+1} = 0, \gamma_{n+2} = 0, \dots$ *for some* $n \geq 0$.

The latter occurs if and only if $f \in \mathscr{B}_n$. Conversely, for any sequence $\overrightarrow{\gamma} = (\gamma_0, \gamma_1, \dots)$ satisfying one of the above conditions, there exists a unique function $f \in \mathscr{S}$ such that $\overrightarrow{\gamma}(f) = \overrightarrow{\gamma}$.

We denote by X_0 the set of vectors $\overrightarrow{\gamma} = (\gamma_0, \gamma_1, \dots)$ satisfying one of the above two conditions. Then $\mathscr{S}$ can be identified with X_0 as a set. Here, we briefly explain an idea of the proof of Schur's theorem (see Wall [4] for details).

For convenience of the reader, we give an outline of the proof of Schur's theorem. Let $f \in \mathscr{S}$ and define f_j $(j = 0, 1, 2, \dots)$ as before. By definition,

$$f_{j+1}(z) = \frac{1}{z} \cdot \frac{f_j(z) - \gamma_j}{1 - \bar{\gamma}_j f_j(z)},$$

which can be rewritten as

$$\begin{aligned} f_j(z) &= \frac{z f_{j+1}(z) + \gamma_j}{1 + \bar{\gamma}_j z f_{j+1}(z)} \\ &= \gamma_j + \frac{(1 - |\gamma_j|^2) z f_{j+1}(z)}{\bar{\gamma}_j z f_{j+1}(z) + 1} \\ &= \gamma_j + \cfrac{(1 - |\gamma_j|^2) z}{\bar{\gamma}_j z + \cfrac{1}{f_{j+1}(z)}}. \end{aligned}$$

By a repated use of this, we arrive at the Schur continued fraction expansion, which converges locally uniformly on the unit disk $\mathbb{D}$:

$$f(z) = \gamma_0 + \cfrac{(1 - |\gamma_0|^2) z}{\bar{\gamma}_0 z + \cfrac{1}{\gamma_1 + \cfrac{(1 - |\gamma_1|^2) z}{\bar{\gamma}_1 z + \cfrac{1}{\gamma_2 + \ddots}}}}.$$

Let us see how to show it in more detail. For simplicity, we consider only a generic case; namely, $|\gamma_j| < 1$ for all j. We denote by T_j the Möbius transformation represented by the matrix

$$A_j = \begin{pmatrix} z & \gamma_j \\ \bar{\gamma}_j z & 1 \end{pmatrix}, \quad j = 0, 1, 2, \dots.$$

Namely, $T_j(w) = (zw + \gamma_j)/(\bar{\gamma}_j zw + 1)$ and $T_j(\mathbb{D}) \subset \mathbb{D}$ for each $z \in \mathbb{D}$. Then

$$f_j(z) = \frac{z f_{j+1}(z) + \gamma_j}{1 + \bar{\gamma}_j f_{j+1}(z)} = T_j(f_{j+1}(z)).$$

Hence,

$$f(z) = (T_0 \circ T_1 \circ \dots \circ T_j)(f_{j+1}(z)) = U_j(f_{j+1}(z)),$$

where $U_j = T_0 \circ T_1 \circ \cdots \circ T_j$. The Möbius transformation U_j is represented by the matrix

$$B_j = A_0 \cdots A_j = \begin{pmatrix} p_j(z) & q_j(z) \\ r_j(z) & s_j(z) \end{pmatrix}.$$

The truncated continued fraction is expressed by $F_j(z) := U_j(0) = q_j(z)/s_j(z)$. Since $\det A_j = (1 - |\gamma_j|^2)z$, we have

$$\det B_j = z^{j+1} \prod_{k=0}^{j} (1 - |\gamma_k|^2).$$

Then, by the formula

$$U(w_1) - U(w_2) = \frac{(ad - bc)(w_1 - w_2)}{(cw_1 + d)(cw_2 + d)}$$

for $U(w) = (aw + b)/(cw + d)$,

$$f(z) - F_j(z) = U_j(f_{j+1}(z)) - U_j(0) = \frac{f_{j+1}(z) z^{j+1} \prod_{k=0}^{j}(1 - |\gamma_k|^2)}{\{r_j(z) f_{j+1}(z) + s_j(z)\} s_j(z)}.$$

On the other hand,

$$|f(z) - F_j(z)| \le |f(z)| + |F_j(z)| \le 2.$$

Thus (a slightly extended) Schwarz Lemma now yields

$$|f(z) - F_j(z)| \le 2|z|^{j+1}.$$

Thus the truncated continued fraction converges to $f(z)$:

$$F_j(z) = \gamma_0 + \cfrac{(1 - |\gamma_0|^2)z}{\bar{\gamma}_0 z + \cfrac{1}{\gamma_1 + \cfrac{(1 - |\gamma_1|^2)z}{\ddots + \cfrac{}{\cdots + \gamma_j}}}} \to f(z).$$

Conversely, if we are given a sequence γ_j $(j = 0, 1, 2, \ldots)$ with $|\gamma_j| < 1$ we can construct f as a limit of the functions F_j defined as the truncated continued fraction. In this way, we consruct a function $f \in \mathscr{S}$ which has γ_j $(j = 0, 1, 2, \ldots)$ as its Schur parameters.

3. Topology of the Schur class

The Schur class $\mathscr{S}$ has the topology of uniform convergence on each compact subset of $\mathbb{D}$. Then $\mathscr{S}$ becomes a compact separable metrizable space. We would like to understand this topology in terms of the Schur parameters. By Schur's theorem, we can identify $\mathscr{B}_d$ with the set

$$\{(\gamma_0, \ldots, \gamma_d, 0, 0, \ldots) : |\gamma_j| < 1 \ (j = 0, \ldots, d-1), |\gamma_d| = 1\} = \mathbb{D}^d \times \mathbb{T} \times 0,$$

and thus $\mathscr{S}$ can be regarded as the set

$$X_0 = \mathbb{D}^{\mathbb{N}_0} \cup \bigcup_{d=0}^{\infty} (\mathbb{D}^d \times \mathbb{T} \times 0).$$

However, the topology of X_0 inherited from $\mathscr{S}$ is different from the relative topology in $\overline{\mathbb{D}}^{\mathbb{N}_0}$. At this point, the following statement is almost clear.

Proposition 3.1. *$\mathscr{B}_d$ is homeomorphic to $\mathbb{D}^d \times \mathbb{T}$.*

Remark 3.2. *In the recent book [1], Garcia, Mashreghi and Ross describe the space, say $\mathscr{B}_d^0$, of Blaschke products f of degree d with $f(1) = 1$ as the symmetric quotient of the set of d-tuples of zeros of $f(z)$ or the set of d-tuples of the critical points of $zf(z)$. These are topologically the d-fold symmetric product of $\mathbb{D}$, which is known to be homeomorphic to $\mathbb{D}^d$. Hence we have again the same topological description $\mathscr{B}_d \cong \mathscr{B}_d^0 \times \mathbb{T} \cong \mathbb{D}^d \times \mathbb{T}$.*

We define an equivalence relation in $\overline{\mathbb{D}}^{\mathbb{N}_0}$ as follows. Two vectors $\vec{\gamma} = (\gamma_0, \gamma_1, \dots)$ and $\vec{\delta} = (\delta_0, \delta_1, \dots)$ in $\overline{\mathbb{D}}^{\mathbb{N}_0}$ are said to be equivalent and written as $\vec{\gamma} \sim \vec{\delta}$ if either $\vec{\gamma} = \vec{\delta}$, or there is $n \in \mathbb{N}_0$ such that $\gamma_j = \delta_j \in \mathbb{D}$ for $j = 0, 1, \dots, n-1$ and that $\gamma_n = \delta_n \in \mathbb{T}$.

Let X be the set of all the equivalence classes $[\vec{\gamma}]$, $\vec{\gamma} \in \overline{\mathbb{D}}^{\mathbb{N}_0}$, and let $\pi : \overline{\mathbb{D}}^{\mathbb{N}_0} \to X$ be the canonical projection: $\pi(\vec{\gamma}) = [\vec{\gamma}]$. Let X be equipped with the quotient topology so that π is a continuous open mapping. Note that the restriction $\pi : X_0 \to X$ is bijective. Then we have the following result.

Theorem 3.3. *X is homeomorphic to $\mathscr{S}$.*

It is, however, still not clear how $\tilde{\mathscr{B}}_d$ is embedded in X. The construction of X is rather artificial. In what follows, we will construct a more natural realization of the quotient map π. To this end, we define

$$\gamma^* = \sqrt{1 - |\gamma|^2}$$

for $\gamma \in \overline{\mathbb{D}}$. Then, for $\vec{\gamma} = (\gamma_0, \gamma_1, \dots) \in \overline{\mathbb{D}}^{\mathbb{N}_0}$, we set

$$(x_0, x_1, \dots) = E(\vec{\gamma}) = (\gamma_0, \gamma_0^*\gamma_1, \gamma_0^*\gamma_1^*\gamma_2, \dots). \tag{3.1}$$

More precisely, $x_0 = \gamma_0$ and

$$x_n = \gamma_n \prod_{j=0}^{n-1} \gamma_j^*$$

for $n = 1, 2, \dots$. The following can be verified easily by an induction argument.

Lemma 3.4.

$$\sum_{j=0}^{n} |x_j|^2 = \sum_{j=0}^{n} |\gamma_0^*\gamma_1^* \cdots \gamma_{j-1}^*\gamma_j|^2 = 1 - \prod_{j=0}^{n} \left(\gamma_j^*\right)^2.$$

In particular, we have

$$\|E(\vec{\gamma})\|_2^2 = 1 - \prod_{n=0}^{\infty}(1 - |\gamma_n|^2)$$

Here, for $\vec{x} = (x_0, x_1, \dots)$,

$$\|\vec{x}\|_2 = \sqrt{\sum_{n=0}^{\infty} |x_n|^2}.$$

We denote by Y the closed unit ball of the space $\ell^2 = \ell^2(\mathbb{N}_0) = \{\vec{x} \in \mathbb{C}^{\mathbb{N}_0} : \|\vec{x}\|_2 < +\infty\}$. Then the mapping E defined in (3.1) can be regarded as a map from $\overline{\mathbb{D}}^{\mathbb{N}_0}$ into Y. The following can be verified easily.

Proposition 3.5. *For $\vec{\gamma}, \vec{\delta} \in \overline{\mathbb{D}}^{\mathbb{N}_0}$, $E(\vec{\gamma}) = E(\vec{\delta})$ iff $\vec{\gamma} \sim \vec{\delta}$.*

For further properties of the mapping E, we show the following lemma.

Lemma 3.6. *Suppose that a vector $(x_0, \ldots, x_n) \in \mathbb{C}^{n+1}$ satisfies the inequality*

$$\sum_{j=0}^{n-1} |x_j|^2 < 1.$$

Then there is a vector $(\gamma_0, \ldots, \gamma_n) \in \mathbb{D}^n \times \mathbb{C}$ such that

$$x_j = \gamma_0^* \cdots \gamma_{j-1}^* \gamma_j \quad (j = 0, 1, \ldots, n).$$

Proof. We show by induction on n. When $n = 0$ the assertion is clear. We assume that the assertion is true up to n and show that the assertion is true for $n+1$. If $|x_0|^2 + \cdots + |x_n|^2 < 1$, by induction assumption, we have $\gamma_0, \ldots, \gamma_n$ as above. By Lemma,

$$1 - \prod_{j=0}^{n} \left(\gamma_j^*\right)^2 = \sum_{j=0}^{n} |\gamma_0^* \gamma_1^* \cdots \gamma_{j-1}^* \gamma_j|^2 = \sum_{j=0}^{n} |x_j|^2 < 1,$$

which implies $\gamma_0^* \cdots \gamma_n^* \neq 0$. Hence, we can set

$$\gamma_{n+1} = \frac{x_{n+1}}{\gamma_0^* \cdots \gamma_n^*}.$$

Then $(\gamma_0, \ldots, \gamma_{n+1})$ satisfies the required conditions. □

As a consequence of the previous lemma, for a point $(x_0, \ldots, x_n) \in \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ with $x_n \neq 0$, we can construct a vector $(\gamma_0, \ldots, \gamma_n)$ such that

$$x_j = \gamma_0^* \cdots \gamma_{j-1}^* \gamma_j \quad (j = 0, 1, \ldots, n).$$

Then, by Lemma 3.6,

$$1 = \sum_{j=0}^{n} |x_j|^2 = \sum_{j=0}^{n} |\gamma_0^* \gamma_1^* \cdots \gamma_{j-1}^* \gamma_j|^2 = 1 - \prod_{j=0}^{n} |\gamma_j^*|^2,$$

and thus $\gamma_n^* = 0$, which means $\gamma_n \in \mathbb{T}$. Here, we used the fact that $x_n \neq 0$ implies that $\gamma_j \in \mathbb{D}$ for $j = 0, 1, \ldots, n-1$. Therefore we have shown $E(\gamma_0, \ldots, \gamma_n, 0, 0, \ldots) = (x_0, \ldots, x_n, 0, 0, \ldots)$. We summarize as follows.

Lemma 3.7. $E(\overline{\mathbb{D}}^d \times \mathbb{T} \times 0) = \mathbb{S}^{2d+1}$.

We next consider $\vec{x} = (x_0, x_1, \ldots)$ with $\sum_{j=0}^{n} |x_j|^2 < 1$ for any n. Then by the proposition above, we can construct a sequence $\vec{\gamma} = (\gamma_0, \gamma_1, \ldots)$ such that $E(\vec{\gamma}) = \vec{x}$. This, together with the observation in the previous slide, means that the mapping $E : X_0 \to Y$ is surjective. (Recall that Y is the closed unit ball of $\ell^2(\mathbb{N}_0)$.) Finally, we obtain the next result. The proof will appear in a forthcoming paper. It says that E is a realization of the projection $\pi : \overline{\mathbb{D}}^{\mathbb{N}_0} \to X$.

Theorem 3.8 (Main Theorem)**.** *The mapping* $E : \overline{\mathbb{D}}^{\mathbb{N}_0} \to Y = \{\vec{x} \in \ell^2(\mathbb{N}_0) : \|\vec{x}\|_2 \leq 1\}$ *is surjective, open and continuous, where* Y *is equipped with weak-*$*$ *topology of* ℓ^2*. In particular, the mapping* $f \mapsto E(\vec{\gamma}(f))$ *gives a homeomorhism from* $\mathscr{S}$ *to* Y*.*

Proof of Theorem 1.3. The topology of $\tilde{\mathscr{B}}_d$ is same as the relative topology in $\mathscr{S}$. Recall that the Schur vectors of $\tilde{\mathscr{B}}_d$ form the set $\overline{\mathbb{D}}^d \times \mathbb{T} \times 0$. Since $E(\overline{\mathbb{D}}^d \times \mathbb{T} \times 0) = \mathbb{S}^{2d+1}$ by Lemma 3.7, we now see that $\tilde{\mathscr{B}}_d$ is homeomorphic to $\mathbb{S}^{2d+1}$ by the main theorem. □

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Integral operators on the Dirichlet-type spaces

Shûichi Ohno

1 Introduction

Throughout this article let $\mathbb{D}$ be the open unit disk in the complex plane and $\mathcal{H}(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. For a fixed function $\varphi \in \mathcal{H}(\mathbb{D})$, we define two types of integral operators on $\mathcal{H}(\mathbb{D})$:

$$S_\varphi f(z) = \int_0^z \varphi(\zeta) f'(\zeta)\, d\zeta$$

and

$$T_\varphi f(z) = \int_0^z \varphi'(\zeta) f(\zeta)\, d\zeta.$$

The bilinear operator $(f, g) \to \int f\ g'$ was introduced by Calderón in harmonic analysis in the 60's [3]. After his research on commutators of singular integral operators, Pommerenke was probably the first author to consider the boundedness of the operator T_φ on the Hardy space in late 70's. A systematic study of T_φ in late 90's was initiated by Aleman and Siskakis. See surveys [1, 2, 9, 10] for more background and results on T_φ.

We will consider these integral operators on the following space. For $0 < p < \infty$ and $-1 < \alpha < \infty$, let $\mathfrak{D}^p_\alpha$ denote the Dirichlet-type space of all functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_\alpha = |f(0)|^p + (1+\alpha) \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^\alpha\, dA(z) < \infty,$$

where $dA(z) = dxdy/\pi$ denotes the Lebesgue area measure on $\mathbb{D}$.

The space $\mathfrak{D}^2_0$ is the classical Dirichlet space and $\mathfrak{D}^2_1$ is the Hardy-Hilbert space. If $\alpha = p-2$, then $\mathfrak{D}^p_{p-2}$ is the Besov space. For each p, the range of values of the parameter α for which $\mathfrak{D}^p_\alpha$ is most interesting is $p-2 \le \alpha \le p-1$. If $\alpha > p-1$, then it holds that $\mathfrak{D}^p_\alpha = A^p_{\alpha-p}$. On the other hand, if $\alpha < p-2$, then $\mathfrak{D}^p_\alpha \subset H^\infty$.

The Carleson measures for the Dirichlet-type spaces have been studied by some rserachers. In particular, the case $\alpha = p-1$ has actively been investigated ([5, 6]). The space $\mathfrak{D}^p_{p-1}$ is the closest to the Hardy space H^p. If $p \ge 2$, then $H^p \subset \mathfrak{D}^p_{p-1}$ by a classical result due to Littlewood and Paley ([7]) and if $0 < p \le 2$, then $\mathfrak{D}^p_{p-1} \subset H^p$. (For example, see [4].)

We need the next space to characterize properties of integral operators. For $0 < p < \infty$, let B^p denote the Besov-type space of all functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{B^p}^p = |f(0)|^p + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2}(1-|\varphi_\alpha(z)|^2)\, dA(z) < \infty,$$

where $\varphi_\alpha(z) = (a-z)/(1-\overline{a}z)$.

Obviously, B^p is a Möbius invariant subspace of $\mathfrak{D}^p_{p-1}$. Let B^p_o be the space consisting of all functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2}(1-|\varphi_\alpha(z)|^2)\, dA(z) = 0.$$

For nonnegative quantities X and Y, we use the abbreviation $X \lesssim Y$ or $Y \gtrsim X$ which means $X \le CY$ for some inessential constant C. Also, we write $X \approx Y$ whenever $X \lesssim Y \lesssim X$.

2 The main results

We here characterize the boundedness and compactness of S_φ and T_φ on the Dirichlet-type spaces $\mathfrak{D}^p_{p-1}$.

At first we consider the boundedness of S_φ and T_φ.

Theorem 2.1 *For $0 < p < \infty$, S_φ is bounded on $\mathfrak{D}^p_{p-1}$ if and only if $\varphi \in H^\infty$.*

Proof. Suppose $\varphi \in H^\infty$. Then, for $f \in \mathfrak{D}^p_{p-1}$, we have

$$\begin{aligned}\|S_\varphi f\|_{p-1}^p &= p\int_{\mathbb{D}} |\varphi(z)f'(z)|^p (1-|z|^2)^{p-1} dA(z)\\ &\lesssim \|\varphi\|_\infty^p \|f\|_{p-1}^p.\end{aligned}$$

Conversely, we here show only the case $0 < p < 3$. For $a \in \mathbb{D}$, let

$$f_a'(z) = \left(\frac{(1-|a|^2)^2}{(1-\overline{a}z)^4}\right)^{1/p}.$$

Then

$$\begin{aligned}\|f_a\|_{p-1}^p &= (1-|a|^2)^2 + p\int_{\mathbb{D}} \frac{(1-|a|^2)^2}{|1-\overline{a}z|^4}(1-|z|^2)^{p-1} dA(z)\\ &\lesssim (1-|a|^2)^2 + \frac{(1-|a|^2)^2}{(1-|a|^2)^{3-p}}\\ &\lesssim (1-|a|^2)^2 + (1-|a|^2)^{p-1}.\end{aligned}$$

$$
\begin{aligned}
\|S_\varphi f_a\|_{p-1}^p &= p\int_{\mathbb{D}} |\varphi(z) f_a'(z)|^p (1-|z|^2)^{p-1}\, dA(z)\\
&= p\int_{\mathbb{D}} |\varphi(z)|^p \frac{(1-|a|^2)^2}{|1-\overline{a}z|^4}(1-|z|^2)^{p-1} dA(z)\\
&= p\int_{\mathbb{D}} |\varphi(\varphi_a(z))|^p \left(\frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}\right)^{p-1} dA(z)\\
&\geq (1-|a|^2)^{p-1}|\varphi(a)|^p,
\end{aligned}
$$

where the last inequality holds by the subharmonic property of functions ([11, p.73, Lemma 4.12]).

By the boundedness of S_φ,

$$\|S_\varphi f_a\|_{p-1}^p \lesssim \|f_a\|_{p-1}^p.$$

So

$$(1-|a|^2)^{p-1}|\varphi(a)|^p \lesssim (1-|a|^2)^2 + (1-|a|^2)^{p-1}$$

and we have $|\varphi(z)| \leq C$.

Next we will consider the boundedness of T_φ.

Theorem 2.2 *For $0 < p \leq 2$, T_φ is bounded on $\mathfrak{D}_{p-1}^p$ if and only if $\varphi \in B^p$.*

Proof. For $a \in \mathbb{D}$, let

$$f(z) = \left(\frac{1-|a|^2}{(1-\overline{a}z)^2}\right)^{1/p}.$$

Then $f_n \in \mathfrak{D}_{p-1}^p$ and $\|f_n\|_{p-1} \lesssim 1$. Then

$$
\begin{aligned}
\|T_\varphi f\|_{p-1}^p &= \int_{\mathbb{D}} |\varphi'(z)|^p \frac{1-|a|^2}{|1-\overline{a}z|^2}(1-|z|^2)^{p-1} dA(z)\\
&= \int_{\mathbb{D}} |\varphi'(z)|^p (1-|z|^2)^{p-2}(1-|\alpha_a(z)|^2) dA(z).
\end{aligned}
$$

So we have $\varphi \in B^p$. This implication holds for the case $0 < p < \infty$.

Conversely, assume $\varphi \in B^p$. By [11, p.263, Corollary 9.13], $d\mu(z) = |\varphi'(z)|^p(1-|z|^2)^{p-1}dA(z)$ is a Carleson measure. That is, the inclusion mapping from H^p into $L^p(\mathbb{D}, d\mu)$ is bounded for $0 < p < \infty$. Moreover, the inclusion mapping from $\mathfrak{D}_{p-1}^p$ into H^p is bounded for $0 < p \leq 2$. Consequently T_φ is bounded on $\mathfrak{D}_{p-1}^p$.

We consider the compactness. To characterize the compactness, we need the following "weak convergence theorem", which is easily proved by the normal family argument.

Proposition 2.3 *Let $T = S_\varphi$ or T_φ for analytic function φ on $\mathbb{D}$. For $0 < p < \infty$, suppose that T is bounded on $\mathfrak{D}_{p-1}^p$. Then T is compact on $\mathfrak{D}_{p-1}^p$ if and only if for any bounded sequence $\{f_n\}$ in $\mathfrak{D}_{p-1}^p$ that converges to 0 uniformly on every compact subset of $\mathbb{D}$, $\|T_\varphi f_n\|_{p-1}$ converges to 0.*

Theorem 2.4 *Let $0 < p < \infty$ and suppose that S_φ is bounded on $\mathfrak{D}^p_{p-1}$. Then S_φ is compact on $\mathfrak{D}^p_{p-1}$ if and only if $\varphi \equiv 0$.*

Proof. We show only the case $0 < p \le 1$. For $\lambda_n \in \mathbb{D}$ with $|\lambda_n| \to 1$ as $n \to \infty$, let

$$f_n(z) = \frac{p}{(4-p)\overline{\lambda_n}} \frac{(1-|\lambda_n|^2)^{(3-p)/p}}{(1-\overline{\lambda_n}z)^{(4-p)/p}}.$$

Then $f_n \in \mathfrak{D}^p_{p-1}$, $\|f_n\|_{p-1}$ is bounded and f_n converges to 0 uniformly on any compact subset of $\mathbb{D}$. Then, by the compactness of S_φ,

$$\|S_\varphi f_n\|_{p-1} \to 0 \quad \text{as} \quad n \to \infty.$$

That is,

$$\begin{aligned}\|S_\varphi f_n\|^p_{p-1} &= \int_{\mathbb{D}} |\varphi(z)|^p \frac{(1-|\lambda_n|^2)^{3-p}}{|1-\overline{\lambda_n}z|^4}(1-|z|^2)^{p-1}dA(z) \\ &= \int_{\mathbb{D}} \frac{|\varphi(\varphi_{\lambda_n}(z))|^p}{|1-\overline{\lambda_n}z|^{2(p-1)}}(1-|z|^2)^{p-1}dA(z) \\ &\ge |\varphi(\lambda_n)|^p.\end{aligned}$$

So we have $\varphi \equiv 0$.

Theorem 2.5 *Let $0 < p \le 2$ and suppose that T_φ is bounded on $\mathfrak{D}^p_{p-1}$. Then T_φ is compact on $\mathfrak{D}^p_{p-1}$ if and only if $\varphi \in B^p_0$.*

Proof. For $\lambda_n \in \mathbb{D}$ with $|\lambda_n| \to 1$ as $n \to \infty$, let

$$f_n(z) = \left(\frac{1-|\lambda_n|^2}{(1-\overline{\lambda_n}z)^2}\right)^{1/p}.$$

Then $f_n \in \mathfrak{D}^p_{p-1}$, $\|f_n\|_{p-1}$ is bounded and f_n converges to 0 uniformly on any compact subset of $\mathbb{D}$. Then, by the compactness of T_φ,

$$\|T_\varphi f_n\|_{p-1} \to 0 \quad \text{as} \quad n \to \infty.$$

That is,

$$\begin{aligned}\|T_\varphi f_n\|^p_{p-1} &= \int_{\mathbb{D}} |\varphi'(z)|^p \frac{1-|\lambda_n|^2}{|1-\overline{\lambda_n}z|^2}(1-|z|^2)^{p-1}dA(z) \\ &= \int_{\mathbb{D}} |\varphi'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_a(z)|^2)dA(z).\end{aligned}$$

So we have $\varphi \in B^p_0$. This implication is true in the case $0 < p < \infty$.

Conversly, assume $\varphi \in B^p_0$. By [11, Theorem 9.14], $d\mu(z) = |\varphi'(z)|^p(1-|z|^2)^{p-1}dA(z)$ is a vanishing Carleson measure. That is, the inclusion mapping from H^p into $L^p(\mathbb{D}.\mu)$ is compact for $0 < p < \infty$. Moreover, the inclusion mapping from $\mathfrak{D}^p_{p-1}$ into H^p is bounded for $0 < p \le 2$. Consequently T_φ is compact on $\mathfrak{D}^p_{p-1}$.

Rättyä ([8]) also considered the boundedness and compactness of T_φ.

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Surjective isometries on a Banach space of analytic functions on the open unit disc

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1 Introduction

$(M, \|\cdot\|_M)$, $(N, \|\cdot\|_N)$ をそれぞれ (複素) ノルム空間とする．

$$\|T(a) - T(b)\|_N = \|a - b\|_M \quad (\forall a, b \in M)$$

を満たすとき，T を $(M, \|\cdot\|_M)$ から $(N, \|\cdot\|_N)$ への等距離写像 (isometry) という．ただし，写像 T には必ずしも複素線形性 (complex linear) を仮定していない．Mazur-Ulam theorem [16] により，ノルム空間からノルム空間への全射等距離写像 T は，$T(0) = 0$ を満たすならば，実線形 (real linear) である．

様々な研究者により，ノルム空間からノルム空間への複素線形等距離写像の研究が行なわれている．また，正則関数からなる Banach 空間上の複素線形等距離写像の研究もたくさんある．我々の結果と比較するため，その一部を紹介する．

$\mathbb{D}$ を複素平面の単位開円板，$\bar{\mathbb{D}}$ を単位閉円板，$\mathbb{T}$ を単位円周とする．$H(\mathbb{D})$ を $\mathbb{D}$ 上の正則関数全体からなる集合とする．

$$H^p = \left\{ f \in H(\mathbb{D}) : ||f||_{H^p} := \sup_{r<1} \left\{ \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right\}^{\frac{1}{p}} < \infty \right\}$$

$1 \le p < \infty$ のとき，$(H^p, ||\cdot||_{H^p})$ は 複素 Banach 空間である．特に，$(H^2, ||\cdot||_{H^2})$ は 複素 Hilbert 空間である．

Theorem 1 (Forelli [7]) p を $1 \le p < \infty$ かつ $p \ne 2$ とする．T が $(H^p, ||\cdot||_{H^p})$ から $(H^p, ||\cdot||_{H^p})$ への全射な複素線形等距離写像とすると，$c \in \mathbb{T}$ と等角写像 $\phi : \mathbb{D} \to \mathbb{D}$ が存在して，

$$Tf(z) = c \cdot (\phi'(z))^{\frac{1}{p}} \cdot f(\phi(z)), \qquad (f \in H^p,\ z \in \mathbb{D})$$

と表す事ができる．逆に，上のように T を定めると，T は $(H^p, ||\cdot||_{H^p})$ から $(H^p, ||\cdot||_{H^p})$ への全射な複素線形等距離写像となる．

$p = 1$ とすると，deLeeue, Rudin and Wermer の結果 [5] が得られる．

$$\mathcal{S}^p := \{f \in H(\mathbb{D}) : f' \in H^p\} \qquad (1 \leq p < \infty)$$

とおく．$\mathcal{S}^p$ には次のようなノルムを定める事ができる．

$$\begin{aligned} ||f||_\sigma &:= |f(0)| + ||f'||_{H^p} \\ ||f||_\Sigma &:= ||f||_\infty + ||f'||_{H^p} \\ &= \sup_{z \in \bar{\mathbb{D}}} |f(z)| + ||f'||_{H^p} \end{aligned}$$

$f' \in H^p$ ならば，f は $\bar{\mathbb{D}}$ へ連続に拡張する事ができるので ([6, Theorem 3.11])，$||f||_\infty = \sup_{z \in \bar{\mathbb{D}}} |f(z)|$ を考える事ができる． $1 \leq p < \infty$ のとき，$(\mathcal{S}^p, ||\cdot||_\sigma)$, $(\mathcal{S}^p, ||\cdot||_\Sigma)$ はそれぞれ複素 Banach 空間である．

Theorem 2 (Novinger and Oberlin [20]) p を $1 \leq p < \infty$ かつ $p \neq 2$ とする．
(a) T が $(\mathcal{S}^p, ||\cdot||_\sigma)$ から $(\mathcal{S}^p, ||\cdot||_\sigma)$ への全射な複素線形等距離写像とすると，$c \in \mathbb{T}$ と等角写像 $\phi : \mathbb{D} \to \mathbb{D}$ が存在して，

$$Tf(z) = c \cdot f(0) + \int_{[0,z]} (\phi'(\zeta))^{\frac{1}{p}} f'(\phi(\zeta))\, d\zeta \qquad (f \in \mathcal{S}^p,\ z \in \mathbb{D})$$

と表す事ができる．逆に，上のように T を定めると，T は $(\mathcal{S}^p, ||\cdot||_\sigma)$ から $(\mathcal{S}^p, ||\cdot||_\sigma)$ への全射な複素線形等距離写像となる．
(b) T が $(\mathcal{S}^p, ||\cdot||_\Sigma)$ から $(\mathcal{S}^p, ||\cdot||_\Sigma)$ への全射な複素線形等距離写像とすると，$c \in \mathbb{T}$ と等角写像 $\phi : \mathbb{D} \to \mathbb{D}$ が存在して，

$$Tf(z) = c \cdot f(\phi(z)) \qquad (f \in \mathcal{S}^p,\ z \in \mathbb{D})$$

と表す事ができる．逆に，上のように T を定めると，T は $(\mathcal{S}^p, ||\cdot||_\Sigma)$ から $(\mathcal{S}^p, ||\cdot||_\Sigma)$ への全射な複素線形等距離写像となる．

$\mathbb{D}$ 上の正則関数であり，$\bar{\mathbb{D}}$ 上へ連続に拡張可能な関数全体を $A(\bar{\mathbb{D}})$ とする．$A(\bar{\mathbb{D}})$ には

$$||f||_\infty := \sup_{z \in \bar{\mathbb{D}}} |f(z)|$$

によりノルムを定める事ができ，$(A(\bar{\mathbb{D}}), ||\cdot||_\infty)$ は複素 Banach 空間である．Novinger and Oberlin にならい，

$$\mathcal{S}_A := \{f \in H(\mathbb{D}) : f' \in A(\bar{\mathbb{D}})\}$$

とおく．$\mathcal{S}_A$ には

$$||f||_\sigma := |f(0)| + ||f'||_\infty$$

によりノルムを定める事ができ，$(\mathcal{S}_A, ||\cdot||_\sigma)$ は複素 Banach 空間である．

$(\mathcal{S}_A, ||\cdot||_\sigma)$ から $(\mathcal{S}_A, ||\cdot||_\sigma)$ への，複素線形とは限らない，全射な等距離写像の形について，次の結果が得られた．

2 Main Result

Theorem 3 T が $(\mathcal{S}_A, ||\cdot||_\sigma)$ から $(\mathcal{S}_A, ||\cdot||_\sigma)$ への全射等距離写像とすると，次の 4 つの内の 1 つの形である．

$c_{1,1}, c_{1,2}, \lambda_1 \in \mathbb{T}$ と $a_1 \in \mathbb{D}$ が存在して

$$T(f)(z) = T(0)(z) + c_{1,1}f(0) + \int_{[0,z]} c_{1,2}f'(\rho(\zeta))\,d\zeta \quad (\forall f \in \mathcal{S}_A,\ \forall z \in \mathbb{D}),$$

$c_{2,1}, c_{2,2}, \lambda_2 \in \mathbb{T}$ と $a_2 \in \mathbb{D}$ が存在して

$$T(f)(z) = T(0)(z) + \overline{c_{2,1}f(0)} + \int_{[0,z]} c_{2,2}f'(\rho(\zeta))\,d\zeta \quad (\forall f \in \mathcal{S}_A,\ \forall z \in \mathbb{D}),$$

$c_{3,1}, c_{3,2}, \lambda_3 \in \mathbb{T}$ と $a_3 \in \mathbb{D}$ が存在して

$$T(f)(z) = T(0)(z) + c_{3,1}f(0) + \int_{[0,z]} \overline{c_{3,2}f'(\rho(\overline{\zeta}))}\,d\zeta \quad (\forall f \in \mathcal{S}_A,\ \forall z \in \mathbb{D}),$$

$c_{4,1}, c_{4,2}, \lambda_4 \in \mathbb{T}$ と $a_4 \in \mathbb{D}$ が存在して

$$T(f)(z) = T(0)(z) + \overline{c_{4,1}f(0)} + \int_{[0,z]} \overline{c_{4,2}f'(\rho(\overline{\zeta}))}\,d\zeta \quad (\forall f \in \mathcal{S}_A,\ \forall z \in \mathbb{D}),$$

ここで，ρ は $z \in \bar{\mathbb{D}}$, $j = 1, 2, 3, 4$ に対して，$\rho(z) = \lambda_j \dfrac{z - a_j}{\overline{a_j}z - 1}$ である．

逆に，T を上の 4 つの内の 1 つの形に定めると，T は $(\mathcal{S}_A, ||\cdot||_\sigma)$ から $(\mathcal{S}_A, ||\cdot||_\sigma)$ への全射等距離写像である．

詳しくは [18] を見て欲しい．

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Bounded subsets of Smirnov and Privalov classes on the upper half plane

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Abstract. In this note, some characterizations of boundedness in $N_*(D)$ and $N^p(D)\,(1 < p < \infty)$ will be described, where $N_*(D)$ denote the Smirnov class and $N^p(D)$ the Privalov class on the upper half plane $D = \{z \in \mathbf{C} \,|\, \mathtt{Im}z > 0\}$, respectively.[1]

Key words: bounded subsets, Privalov class, Smirnov class, Nevanlinna class.

1. Introduction

Let U and T denote the unit disk and the unit circle in $\mathbf{C}$, respectively. The Privalov class $N^p(U)$, $1 < p < \infty$, is defined as the set of all holomorphic functions f on U and satisfying

$$\sup_{0<r<1} \int_T (\log(1 + |f(r\zeta)|))^p \, d\sigma(\zeta) < +\infty,$$

where $d\sigma$ denotes normalized Lebesgue measure on T. The notion of $N^p(U)$ was introduced by Privalov [1], and has been explored by several authors (see [2, 3, 4]). Letting $p = 1$, we have the Nevanlinna class $N(U)$. It is well-known that each function f in $N(U)$ has the nontangential limit $f^*(\zeta) = \lim_{r\to 1^-} f(r\zeta)$ (a.e. $\zeta \in T$) and that $\log(1 + |f|)$ (and hence, $(\log(1 + |f|))^p$ for $p > 1$) is subharmonic if f is holomorphic. Define a metric

$$d_{N^p(U)}(f, g) = \left\{ \int_T (\log(1 + |f^*(\zeta) - g^*(\zeta)|))^p \; d\sigma(\zeta) \right\}^{\frac{1}{p}}$$

for $f, g \in N^p(U)$. With the metric $d_{N^p(U)}(\cdot, \cdot)$ $N^p(U)$ becomes an F-algebra [2]. Recall that an F-algebra is a topological algebra in which the topology arises from a complete metric.

We denote the Smirnov class by $N_*(U)$, which consists of all holomorphic functions f on U such that $\log(1 + |f(z)|) \le Q[\phi](z)$ $(z \in U)$ for some $\phi \in L^1(T), \phi \ge 0$, where the right side denotes the Poisson integral of ϕ on U. It is known that, if $f \in N(U)$, f belongs to $N_*(U)$ if and only if

$$\lim_{r\to 1^-} \int_T \log(1 + |f(r\zeta)|) \, d\sigma(\zeta) = \int_T \log(1 + |f^*(\zeta)|) \, d\sigma(\zeta).$$

Under the metric

$$d_{N_*(U)}(f, g) = \int_T \log(1 + |f^*(\zeta) - g^*(\zeta)|) \, d\sigma(\zeta)$$

for $f, g \in N_*(U)$, the class is also an F-algebra (see [5]).

For $0 < p < \infty$, the class $M^p(U)$ is defined as the set of all holomorphic functions f on U such that

$$\int_T (\log(1 + Mf(\zeta)))^p \, d\sigma(\zeta) < +\infty,$$

[1] *2010 Mathematics Subject Classification* : 30H50, 46E10.

where $Mf(\zeta) = \sup_{0\le r<1} |f(r\zeta)|$ is the maximal function. The class $M^1(U)$ was introduced by Kim in [6]. As for $p > 0$, the class was considered in [7, 8]. For $f, g \in M^p(U)$, define a metric

$$d_{M^p(U)}(f,g) = \left\{ \int_T (\log(1 + M(f-g)(\zeta)))^p \, d\sigma(\zeta) \right\}^{\frac{\alpha_p}{p}},$$

where $\alpha_p = min(1,p)$. With this metric $M^p(U)$ is also an F-algebra (see [9]).

It is well-known that $H^q(U) \subsetneq N^p(U) \subsetneq M^1(U) \subsetneq N_*(U) \subsetneq N(U)$ $(0 < q \le +\infty\,, p > 1)$, where $H^q(U)$ denotes the Hardy space on U. Moreover, it is known that $N(U) \subsetneq M^p(U)$ $(0 < p < 1)$ [6].

Mochizuki [10] introduced the Nevanlinna class $N_0(D)$ and the Smirnov class $N_*(D)$ on the upper half plane $D := \{z \in \mathbf{C} \,|\, \mathtt{Im} z > 0\}$: the class $N_0(D)$ is the set of all holomorphic functions f on D satisfying

$$\sup_{y>0} \int_{\mathbf{R}} \log(1 + |f(x+iy)|)\, dx < +\infty$$

and $N_*(D)$ the set of all holomorphic functions f on D satisfying $\log(1+|f(z)|) \le P[\phi](z)$ $(z \in D)$ for some $\phi \in L^1(\mathbf{R}), \phi \ge 0$, where the right side denotes the Poisson integral of ϕ on D. It is well-known that each function f in $N_0(D)$ has the nontangential limit $f^*(x) = \lim_{y\to 0^+} f(x+iy)$ (a.e. $x \in \mathbf{R}$). Let $f \in N_0(D)$. Then $f \in N_*(D)$ if and only if

$$\lim_{y\to 0^+} \int_{\mathbf{R}} \log(1 + |f(x+iy)|) dx = \int_{\mathbf{R}} \log(1 + |f^*(x)|) dx$$

(see [10]). Moreover, under the metric

$$d_{N_*(D)}(f\,,\,g) = \int_{\mathbf{R}} \log(1 + |f^*(x) - g^*(x)|)\, dx,$$

the class $N_*(D)$ becomes an F-algebra [10].

The class $M^p(D)$ $(0 < p < \infty)$ is defined as the set of all holomorphic functions f on D such that

$$\int_{\mathbf{R}} (\log(1 + Mf(x)))^p \, dx < +\infty,$$

where $Mf(x) = \sup_{y>0} |f(x+iy)|$. The class $M^p(X)$ with $p = 1$ was introduced by Ganzhula in [11]. As for $p > 0$, Efimov and Subbotin investigated this class [12]. For $f, g \in M^p(D)$, define a metric

$$d_{M^p(D)}(f,g) = \left\{ \int_{\mathbf{R}} (\log(1 + M(f-g)(x)))^p \, dx \right\}^{\frac{\alpha_p}{p}},$$

where $\alpha_p = min(1,p)$. With this metric $M^p(D)$ is also an F-algebra (see [11, 12]).

In [13], the class $N^p(D)$ was introduced, analogous to $N^p(U)$; i.e., we denote by $N^p(D)$ $(p > 1)$ the set of all holomorphic functions f on D such that

$$\sup_{y>0} \int_{\mathbf{R}} (\log(1 + |f(x+iy)|))^p \; dx < +\infty.$$

Each $f \in N^p(D)$ has the nontangential limit $f^*(x)$ for a.e.$x \in \mathbf{R}$, and under the metric

$$d_{N^p(D)}(f\,,\,g) = \left\{ \int_{\mathbf{R}} (\log(1 + |f^*(x) - g^*(x)|))^p \; dx \right\}^{\frac{1}{p}},$$

the class $N^p(D)$ becomes an F-algebra [13].

A subset L of a linear topological space A is said to be *bounded* if for any neighborhood V of zero in A there exists a real number α , $0 < \alpha < 1$, such that $\alpha L = \{\alpha f\,;\, f \in L\} \subset V$. Yanagihara characterized bounded subsets of $N_*(U)$ [14]. As for $M^p(U)$ with $p = 1$, Kim described some characterizations of boundedness (see [6]). For $p > 1$, these characterizations were considered by Meštrović [15]. As for $M^p(D)$ with $p = 1$, Ganzhula investigated the properties of boundedness [11] and Efimov characterized bounded subsets of $M^p(D)$ in the case $0 < p < \infty$ [16]. In recent paper [17], the author described bounded subsets of $M^p(U)$ in the case $0 < p < 1$.

The following are previous studies on characterizations of bounded subsets of function spaces on U or D:

Previous studies on characterizations of bounded subsets of function spaces on U

$N^p(U)$ $(1 < p < \infty)$	$M^p(U)$ $(0 < p < 1)$	$M^1(U)$	$M^p(U)$ $(1 < p < \infty)$	$N_*(U)$
Subbotin (1999)	Iida (2017) [17]	Kim (1988)	Meštrović (2014)	Yanagihara (1973)

Previous studies on characterizations of bounded subsets of function spaces on D

$N^p(D)$ $(1 < p < \infty)$	$M^1(D)$	$M^p(D)$ $(0 < p < \infty)$	$N_*(D)$
Iida (2017) [18]	Ganzhula (2000)	Efimov (2007)	Iida (2017) [18]

2. The results

Theorem 2.1. [18] *Let $p > 1$. $L \subset N^p(D)$ is bounded if and only if*

(i) there exists a $K < \infty$ such that

$$\int_{\mathbf{R}} (\log(1 + |f^*(x)|))^p \, dx < K$$

for all $f \in L$;

(ii) for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_E (\log(1 + |f^*(x)|))^p \, dx < \varepsilon, \quad \textit{for all } f \in L,$$

for any measurable set $E \subset \mathbf{R}$ with the Lebesgue measure $|E| < \delta$.

Proof. We follow [16, Theorem 1].

Necessity. Let L be a bounded subset of $N^p(D)$.

(i) For any number $\eta > 0$ there exists an $\alpha = \alpha(\eta)$, $0 < \alpha < 1$, such that

$$(d_{N^p(D)}(\alpha f\,,\, 0))^p = \int_{\mathbf{R}} (\log(1 + \alpha|f^*(x)|))^p \, dx < \eta^p \tag{2.1}$$

for all $f \in L$. Utilizing the inequality $(1 + x)^\alpha \leq 1 + \alpha x$ $(0 < \alpha < 1\,,\, x \geq 0)$, it follows that, from (2.1),

$$\begin{aligned}\int_{\mathbf{R}} (\log(1 + |f^*(x)|))^p \, dx &\leq \int_{\mathbf{R}} \left(\log(1 + \alpha|f^*(x)|)^{\frac{1}{\alpha}}\right)^p dx \\ &= \frac{1}{\alpha^p} \int_{\mathbf{R}} (\log(1 + \alpha|f^*(x)|))^p \, dx \\ &< \left(\frac{\eta}{\alpha}\right)^p = K = \textit{constant}\end{aligned}$$

for all $f \in L$. Therefore, condition (i) holds.

(ii) For any number $\varepsilon > 0$, we take η as $\eta < \varepsilon^{\frac{1}{p}}/2$. Choose a number $\alpha = \alpha(\varepsilon)$, $0 < \alpha < 1$, such that equality (2.1) holds for all $f \in L$. Then for any measurable set $E \subset \mathbf{R}$, using Minkowski's inequality, we have the estimate

$$\begin{aligned}
\int_E (\log(1 + |f^*(x)|))^p \, dx &< \int_E \left(\log\left(\frac{1}{\alpha} + |f^*(x)|\right)\right)^p dx \\
&= \int_E \left(\log\frac{1}{\alpha} + \log(1 + \alpha|f^*(x)|)\right)^p dx \\
&\le \left(\left(|E|\left(\log\frac{1}{\alpha}\right)^p\right)^{\frac{1}{p}} + \left(\int_{\mathbf{R}} (\log(1 + \alpha|f^*(x)|))^p \, dx\right)^{\frac{1}{p}}\right)^p \\
&< \left(|E|^{\frac{1}{p}} \log\frac{1}{\alpha} + \eta\right)^p.
\end{aligned}$$

If we take $\delta > 0$ as $\delta < \varepsilon/(2^p\,(\log(1/\alpha))^p)$, then

$$\begin{aligned}
\int_E (\log(1 + |f^*(x)|))^p \, dx &< \left(\delta^{\frac{1}{p}} \log\frac{1}{\alpha} + \frac{\varepsilon^{\frac{1}{p}}}{2}\right)^p \\
&< \left(\frac{\varepsilon^{\frac{1}{p}}}{2} + \frac{\varepsilon^{\frac{1}{p}}}{2}\right)^p = \varepsilon
\end{aligned}$$

for all $f \in L$ and any measurable set $E \subset \mathbf{R}$, $|E| < \delta$. Thus condition (ii) holds.

Sufficiency. Let conditions (i) and (ii) hold for a subset L of $N^p(D)$, $p > 1$. Consider a neighborhood

$$V = \{g \in N^p(D) : d_{N^p(D)}(g\,,\,0) < \eta\}.$$

Take $\varepsilon > 0$ as $\varepsilon < \eta^p/3$. According to (ii), there exists a number $\delta > 0$ such that

$$\int_E (\log(1 + |f^*(x)|))^p \, dx < \varepsilon < \frac{\eta^p}{3} \tag{2.2}$$

for all $f \in L$ and any measurable set $E \subset \mathbf{R}$, $|E| < \delta$. Next there exists a finite constant $K > 0$ such that condition (i) holds for all $f \in L$. Applying Chebyshev's inequality to the Lebesgue measure of the set $E_f = \{x \in \mathbf{R} \,|\, (\log(1 + |f^*(x)|))^p > K/\delta\}$ for $f \in L$, the following estimate is valid:

$$|E_f| \le \frac{\delta}{K} \int_{\mathbf{R}} (\log(1 + |f^*(x)|))^p \; dx < \delta.$$

Then we may assume $E = E_f$ and $|f^*(x)| > \exp(K/\delta)^{\frac{1}{p}} - 1 = C$ in inequality (2.2), that is, $|f^*(x)|/C < 1$ for all $x \in \mathbf{R} \backslash E_f$. Therefore, for any number $\alpha\,(0 < \alpha < 1)$ and all $f \in L$, we have the following:

$$\begin{aligned}
&\int_{\mathbf{R}} (\log(1 + \alpha|f^*(x)|)^p \, dx \\
&\quad = \int_{E_f} (\log(1 + \alpha|f^*(x)|))^p \, dx + \int_{\mathbf{R}\backslash E_f} (\log(1 + \alpha|f^*(x)|))^p \, dx \\
&\quad < \int_{E_f} (\log(1 + |f^*(x)|))^p \, dx \qquad (2.3)
\end{aligned}$$

$$+\int_{E_1} (\log(1+\alpha|f^*(x)|))^p\,dx + \int_{E_2} (\log(1+\alpha|f^*(x)|))^p\,dx,$$

where $\mathbf{R}\backslash E_f = E_1 \cup E_2$, $E_1 = \{x \in \mathbf{R} \,|\, |f^*(x)| < 1\}$ and $E_2 = \{x \in \mathbf{R} \,|\, 1 \leq |f^*(x)| < C\}$. By using the elementary inequality $1+\beta t \leq (1+t)^{2\beta}$ $(0 \leq t < 1, 0 < \beta < 1/2)$ to the second integral in (2.3), using (2.2) and taking

$$\alpha = min\left(\frac{1}{2}, \frac{1}{2}\left(\frac{\eta^p}{3K}\right)^{\frac{1}{p}}, \frac{1}{C}\left(2^{\left(\frac{\eta^p}{3K}\right)^{\frac{1}{p}}} - 1\right)\right),$$

we have the following estimate

$$\begin{aligned}
&\int_{\mathbf{R}} (\log(1+\alpha|f^*(x)|))^p\,dx \\
&\quad < \frac{\eta^p}{3} + (2\alpha)^p K + \frac{\eta^p}{3K}\int_{E_2} (\log(1+1))^p\,dx \\
&\quad \leq \frac{\eta^p}{3} + \frac{\eta^p}{3} + \frac{\eta^p}{3K}\int_{\mathbf{R}} (\log(1+|f^*(x)|))^p\,dx \\
&\quad < \eta^p.
\end{aligned}$$

Therefore, $\alpha L \subset V$ and the set L is bounded in $N^p(D)$ by definition.

The proof of the theorem is complete.

□

Next we consider some characterizations of boundedness in $N_*(D)$. Proof of the following theorem can be obtained by taking $p = 1$ in the whole proof of Theorem 2.1; therefore, this proof may be omitted.

Theorem 2.2. [18] *$L \subset N_*(D)$ is bounded if and only if*

(i) there exists a $K < \infty$ such that

$$\int_{\mathbf{R}} \log(1+|f^*(x)|)\,dx < K$$

for all $f \in L$;

(ii) for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_E \log(1+|f^*(x)|)\,dx < \varepsilon, \quad \textit{for all } f \in L,$$

for any measurable set $E \subset \mathbf{R}$ with the Lebesgue measure $|E| < \delta$.

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2-local isometries on C^1

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For a Banach space $\mathcal{B}$, a mapping $T : \mathcal{B} \to \mathcal{B}$ is called a 2*-local isometry* if for each $f, g \in \mathcal{B}$ there exists a surjective complex-linear isometry $T_{f,g} : \mathcal{B} \to \mathcal{B}$ such that $T_{f,g}(f) = T(f)$ and $T_{f,g}(g) = T(g)$. Note that no surjectivity or linearity of T is assumed. In [9], Molnár studied 2-local isometries of the Banach algebra $B(H)$, all bounded linear operators on an infinite separable Hilbert space H. Let $C_0(X)$ be the Banach algebra of all complex-valued continuous functions on a locally compact Hausdorff space X which vanish at infinity equipped with the supremum norm $\|\cdot\|_\infty$. If X is compact, then we write $C(X)$ in stead of $C_0(X)$. In [2], Győry showed that if X is a first countable σ-compact Hausdorff space, then every 2-local isometry on $C_0(X)$ is a surjective complex-linear isometry. In [3], Hatori, Miura, Oka and Takagi showed every 2-local isometry on some uniform algebra is a surjective complex-linear isometry. Jiménez-Vargas and Villegas-Vallecillos [5] considered 2-local isometries on spaces of Lipschitz functions on a bounded separable metric space.

In [4], Hosseini investigated an extension of 2-local isometry. A mapping $T : \mathcal{B} \to \mathcal{B}$ is called a *2-local real-linear isometry* if for each $f, g \in \mathcal{B}$ there exists a surjective real-linear isometry $T_{f,g} : \mathcal{B} \to \mathcal{B}$ such that $T_{f,g}(f) = T(f)$ and $T_{f,g}(g) = T(g)$. No surjectivity or linearity of T is assumed. Let $C^{(n)}([0,1])$ be the Banach space of all n-times continuously differentiable functions on the closed unit interval $[0,1]$ with a norm. For example, $C^{(n)}([0,1])$ with one of the following norms is a Banach space;

$$\|f\|_C = \sup_{t\in[0,1]} \sum_{k=0}^{n} \frac{|f^{(k)}(t)|}{k!},$$

$$\|f\|_\Sigma = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_\infty}{k!},$$

$$\|f\|_\sigma = \sum_{k=0}^{n-1} |f^{(k)}(0)| + \|f^{(n)}\|_\infty,$$

$$\|f\|_m = \max\{|f(0)|, |f'(0)|, \ldots, |f^{(n-1)}(0)|, \|f^{(n)}\|_\infty\},$$

for $f \in C^{(n)}([0,1])$. Hosseini proved that every 2-local real-lienar isometry on $(C^{(n)}([0,1]), \|\cdot\|_m)$ is a surjective real-linear isometry, and showed that if X is a separable, first countable compact Hausdorff space, then every 2-local real-linear isometry on $C(X)$ is a surjective real-linear isometry. Note that Hosseini obtained this result applying the idea which Győry used in [2].

The following theorems are our main results([7, 8]).

Theorem 1. *Every 2-local isometry on* $(C^{(n)}([0,1]), \|\cdot\|_C)$ *is surjective complex-linear isometry.*

Theorem 2. *Every 2-local isometry on* $(C^{(1)}([0,1]), \|\cdot\|_\Sigma)$ *is surjective complex-linear isometry.*

Theorem 3. *Every 2-local real-linear isometry on* $(C^{(1)}([0,1]), \|\cdot\|_\sigma)$ *is surjective real-linear isometry.*

In the proof of these theorems, the characterization of surjective complex-linear (or real-linear) isometries is very important. Put $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $[z]^1 = z$ and $[z]^{-1} = \overline{z}$.

Theorem 4 ([10])**.** *A mapping* T *is a surjective complex-linear isometry on* $(C^{(n)}([0,1]), \|\cdot\|_C)$ *if and only if there exist a unimodular constant* $\lambda \in \mathbb{T}$ *such that* $T(f)(t) = \lambda f(t)$ *for all* $f \in C^{(n)}([0,1])$ *and* $t \in [0,1]$ *or* $T(f)(t) = \lambda f(1-t)$ *for all* $f \in C^{(n)}([0,1])$ *and* $t \in [0,1]$.

Theorem 5 ([1, 11])**.** *A mapping* T *is a surjective complex-linear isometry on* $(C^{(1)}([0,1]), \|\cdot\|_\Sigma)$ *if and only if there exist a unimodular constant* $\lambda \in \mathbb{T}$ *such that* $T(f)(t) = \lambda f(t)$ *for all* $f \in C^{(1)}([0,1])$ *and* $t \in [0,1]$ *or* $T(f)(t) = \lambda f(1-t)$ *for all* $f \in C^{(1)}([0,1])$ *and* $t \in [0,1]$.

Theorem 6 ([6])**.** *A mapping* T *is a surjective real-linear isometry on* $(C^{(1)}([0,1]), \|\cdot\|_\sigma)$ *if and only if there exist* $\epsilon, \delta \in \{\pm 1\}$, *a unimodular constant* $\lambda \in \mathbb{T}$, *a unimodular continuous function* $\beta : [0,1] \to \mathbb{T}$ *and a homeomorphism* $\psi : [0,1] \to [0,1]$ *such that*

$$T(f)(t) = \lambda [f(0)]^\epsilon + \int_0^t \beta(s) [f'(\psi(s))]^\delta \, ds$$

for all $f \in C^{(1)}([0,1])$ *and* $t \in [0,1]$.

Problem. *Is every* 2*-local (real-linear) isometry on* $C^{(n)}([0,1])$ *a surjective complex-linear (or real-linear) isometry?*

space \ norm	C	Σ	σ	m
$C^{(n)}([0,1])$	○ $^{\mathbb{C}}$			○ $^{\mathbb{R}}$
$C^{(1)}([0,1])$	○ $^{\mathbb{C}}$	○ $^{\mathbb{C}}$	○ $^{\mathbb{R}}$	○ $^{\mathbb{R}}$

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Algebraric reflexivity of isometry groups of algebras of Lipschitz maps

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1 Introduction

Let (X,d) be a compact metric space and $(E, \|\cdot\|_E)$ a Banach space. A continuous map $F: X \to E$ is called a Lipschitz map if there exists a positive number L such that

$$\|F(x) - F(y)\|_E \leq Ld(x,y)$$

for every $x,\ y \in X$. For any Lipschitz map F, we define Lipschitz constant $L(F)$ by

$$L(F) = \sup_{x \neq y} \frac{\|F(x) - F(y)\|_E}{d(x,y)}.$$

We denote by $\mathrm{Lip}(X,E)$ the space of all Lipschitz maps from X into E. The space $\mathrm{Lip}(X,E)$ is a Banach space with respect to the max norm $\|\cdot\|_{\max}$,

$$\|F\|_{\max} = \max\{\sup_{x\in X} \|F(x)\|_E, L(F)\}, \quad F \in \mathrm{Lip}(X,E).$$

On the other hand, the space $\mathrm{Lip}(X,E)$ under the sum norm $\|\cdot\|_L$,

$$\|F\|_L = \sup_{x\in X} \|F(x)\|_E + L(F), \quad F \in \mathrm{Lip}(X,E)$$

is a Banach space too. Moreover, if E is a Banach algebra so is $\mathrm{Lip}(X,E)$. For brevity, if no confusion can arise, we write $\|F\|_\infty = \sup_{x\in X}\|F(x)\|_E$. When $E = \mathbb{C}$, we denote $\mathrm{Lip}(X)$ instead of $\mathrm{Lip}(X,\mathbb{C})$. Let M_j be a metric space for $j = 1,2$. We denote the set of all map from M_1 into M_2 by $M(M_1, M_2)$ and the set of all surjective linear isometry from M_1 onto M_2 by $\mathrm{Iso}(M_1, M_2)$. We introduce the definition of local in $\mathrm{Iso}(M_1, M_2)$.

Definition 1. *We say that a bounded linear operator $T: M_1 \to M_2$ is a local in* $\mathrm{Iso}(M_1, M_2)$ *if for any $x \in M_1$, there exists $T_x \in \mathrm{Iso}(M_1, M_2)$ such that*

$$T_x(x) = T(x).$$

The following problem is a main problem for local surjective linear isometries.

Problem 1. *If T is a local in* $\mathrm{Iso}(M_1, M_2)$*, then is T a surjective linear isometry?*

Note that if every local map in $\mathrm{Iso}(M_1, M_2)$ is surjective linear isometry, then we say that $\mathrm{Iso}(M_1, M_2)$ is algebraically reflexive. Botelho and Jamison [1] considered algebraic reflexivity of the isometry group on $(\mathrm{Lip}(X, E), \|\cdot\|_{\max})$ with the additional assumption about X and E by applying a characterization due to Jiménez-Vargas and Villegas-Vallecillos [5] of linear isometries between $\mathrm{Lip}(X, E)$ under the max norm.

In the case of $E = \mathbb{C}$, in [4] Jiménez-Vargas, Morales Campoy and Villegas-Vallecillos proved that every local in $\mathrm{Iso}(\mathrm{Lip}(X), \mathrm{Lip}(X))$ with $\|\cdot\|_L$ is a surjective linear isometry by applying [3, Example 8]. In fact, the statement of [3, Example 8] has been open to the question. The situation is clarified by a recent paper [2, Corollary 15], where a surjective isometry from $\mathrm{Lip}(X_1)$ onto $\mathrm{Lip}(X_2)$ is proved to be of the canonical form as Jarosz and Pathak have described. Hatori and the author [2] proved that a surjective linear isometry between $\mathrm{Lip}(X_j, C(Y_j))$ with the norm $\|\cdot\|_L$ is canonical in the sense that it is a weighted composition operator in [2, Corollary 14].

In addition, the author characterized unital surjective linear isometries on $\mathrm{Lip}(X, M_n(\mathbb{C}))$ with the norm $\|\cdot\|_L$, where $M_n(\mathbb{C})$ is a Banach algebra of complex matrices of degree n with operator norm in [9].

The purpose of this paper is to answer Problem 1 for the case that M_j is $\mathrm{Lip}(X, E)$, which is the algebra of vector-valued Lipschitz maps with respect to $\|\cdot\|_L$ as the norm.

2 Results

Theorem 1 ([6])**.** *Let X_i be a compact metric space for $i = 1, 2$. The set of all surjective linear isometries from* $\mathrm{Lip}(X_1)$ *onto* $\mathrm{Lip}(X_2)$ *is algebraically reflexive.*

This theorem has been proved in [6], but we present a simple proof in [8]. We introduce the sketch of proof as follows.

sketch of proof. Let Ψ be a locally surjective linear isometry from $\mathrm{Lip}(X_1)$ onto $\mathrm{Lip}(X_2)$. Without loss of generality, we may assume $\Psi(1) = 1$. For any $g \neq 0 \in \mathrm{Lip}(X_1)$, we have

$$\Psi(g) = \Psi_g(g) = \alpha_g g \circ \varphi_g,$$

where $\alpha_g \in \mathbb{C}$ with $|\alpha_g| = 1$ and φ_g is a surjective isometry from X_2 onto X_1. There exists $x_0 \in X_1$ such that $|g(x_0)| = \|g\|_\infty$. Let $\lambda \in \mathbb{C}$ with $g(x_0) = \lambda$ with $\lambda \neq 0$. We define $g' \in \mathrm{Lip}(X_1)$ by $g' = g + \lambda 1$. There exists $\alpha_{g'} \in \mathbb{C}$ with $|\alpha_{g'}| = 1$ and a surjective isometry $\varphi_{g'}$ from X_2 onto X_1 such that

$$\begin{aligned}\Psi(g') &= \alpha_{g'} g' \circ \varphi_{g'} = \alpha_{g'}(g + \lambda 1) \circ \varphi_{g'} \\ &= \alpha_{g'} g \circ \varphi_{g'} + \alpha_{g'} \lambda 1,\end{aligned}$$

and

$$\begin{aligned}\Psi(g') &= \Psi(g+\lambda 1)\\ &= \Psi(g)+\Psi(\lambda 1) = \alpha_g g\circ\varphi_g + \lambda 1.\end{aligned}$$

Thus we have

$$\alpha_{g'} g\circ\varphi_{g'} + \alpha_{g'}\lambda 1 = \alpha_g g\circ\varphi_g + \lambda 1. \tag{1}$$

Since $\varphi_{g'}$ is surjective, there exists $x_1 \in X_2$ such that $\varphi_{g'}(x_1) = x_0$. By (1) and $\lambda = g(x_0)$, we have

$$\alpha_{g'}\lambda + \alpha_{g'}\lambda = \alpha_g g(\varphi_g(x_1)) + \lambda. \tag{2}$$

Since $\|g\circ\varphi_g\|_\infty = \|g\|_\infty = |\lambda|$, $\alpha_{g'}, \alpha_g \in \mathbb{T}$, we get $|g(\varphi_g(x_1))| = |\lambda|$. By (2) we obtain

$$|2\alpha_{g'} - 1||\lambda| = |\alpha_g g(\varphi_g(x_1))| = |\lambda|.$$

Since $\lambda \neq 0$, we get $|2\alpha_{g'} - 1| = 1$, hence $\alpha_{g'} = 1$. The equation (1) shows that

$$\sigma(g) \ni g\circ\varphi_{g'}(x) = \alpha_g g\circ\varphi_g(x) = \Psi(g)(x),$$

for any $x \in X_2$, where $\sigma(g)$ denote the spectrum of g. By the Gleason-Kahane-Żelazko theorem, we have Ψ is multiplicative. This implies that $\Psi : \mathrm{Lip}(X_1) \to \mathrm{Lip}(X_2)$ is an algebra homomorphism with $\Psi(1) = 1$. By [10, Theorem 5.1], there is a Lipschitz map $\varphi : X_2 \to X_1$ such that

$$\Psi(g)(x) = g(\varphi(x)), \quad x \in X_2$$

for every $g \in \mathrm{Lip}(X_1)$. Since Ψ is local map, it follows that φ is a surjective isometry. □

Theorem 2 ([2])**.** *Let X_i be a compact metric space and Y_i a compact Hausdorff space for $i = 1, 2$. The map $U : \mathrm{Lip}(X_1, C(Y_1)) \to \mathrm{Lip}(X_2, C(Y_2))$ is a surjective linear isometry if and only if there exists a unimodular function $f \in C(Y_2)$, a continuous map $\varphi : X_2 \times Y_2 \to X_1$ such that $\varphi(\cdot, \phi) : X_2 \to X_1$ is a surjective isometry for any $\phi \in Y_2$, and a homeomorphism $\tau : Y_2 \to Y_1$ which satisfys that*

$$UF(x,\phi) = f(\phi)F(\varphi(x,\phi), \tau(\phi)) \quad x \in X_2, \phi \in Y_2.$$

Applying these theorems, we deduce the next theorem.

Theorem 3 ([8])**.** *Let X_i be a compact metric space and Y_i a compact Hausdorff space for $i = 1, 2$. If the set of all surjective linear isometries from $C(Y_1)$ onto $C(Y_2)$ is algebraically reflexive, then the set of all surjective linear isometries from $\mathrm{Lip}(X_1, C(Y_1))$ onto $\mathrm{Lip}(X_2, C(Y_2))$ is algebraically reflexive.*

It is well known that the set of all surjective linear isometries from $C(Y_1)$ onto $C(Y_2)$ is not always algebraically reflexive. (see [7].)

Theorem 4 ([9]). *Let X_j be a compact metric space for $j = 1, 2$. Then $U : \mathrm{Lip}(X_1, M_n(\mathbb{C})) \to \mathrm{Lip}(X_2, M_n(\mathbb{C}))$ is a linear surjective isometry such that $U(\mathbf{1}) = \mathbf{1}$ if and only if there exists a unitary matrix $V \in M_n(\mathbb{C})$, and a surjective isometry $\varphi : X_2 \to X_1$, such that*

$$(UF)(x) = VF(\varphi(x))V^{-1}, \quad F \in \mathrm{Lip}(X_1, M_n(\mathbb{C})),\ x \in X_2$$

or

$$(UF)(x) = VF^t(\varphi(x))V^{-1}, \quad F \in \mathrm{Lip}(X_1, M_n(\mathbb{C})),\ x \in X_2,$$

where $F^t(y)$ denote transpose of $F(y)$ for $y \in X_1$.

Theorem 5 ([8]). *Let X_i be a compact metric space for $i = 1, 2$. The set of all unital surjective linear isometries from* $\mathrm{Lip}(X_1, M_n(\mathbb{C}))$ *onto* $\mathrm{Lip}(X_2, M_n(\mathbb{C}))$ *is algebraically reflexive.*

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2-LOCAL SURJECTIVE ISOMETRIES ON SOME SPACES OF CONTINUOUS FUNCTIONS

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Abstract. We study the group of all surjective isometries of the Banach algebra of continuously differentiable functions from the point of view of how they are determined by their local actions.

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Let X be a linear space and $L(X)$ the set of all linear maps on X. Suppose that $\emptyset \neq S \subset L(X)$.

Definition 1. Let $T \in L(X)$. We say that T is local in S, if for every $x \in X$ there exists $T_x \in S$ which satisfies that

$$T(x) = T_x(x).$$

The study of Local map dates back to the seminal work of Kadison, and Larsen and Sourour. Motivated by an interesting extension by Kowalski and Słodkowski of the Gleason-Kahane-Żelazko theorem, Šemrl [15] initiated to study 2-local automorphisms and derivations. At the cost of requiring the local behavior at every two points, the condition of linearity is dropped.

Let $\mathcal{X}$ be a non-empty set. Let $\mathcal{M}(\mathcal{X})$ be the set of all maps on $\mathcal{X}$. Suppose that $\emptyset \neq S \subset \mathcal{M}(\mathcal{X})$.

Definition 2. Let $T \in \mathcal{M}(\mathcal{X})$. We say that T is 2-local in S if for every pair $x, y \in \mathcal{X}$ there exists $T_{x,y} \in S$ such that

$$T(x) = T_{x,y}(x), \quad T(y) = T_{x,y}(y).$$

If every 2-local map in S is in fact an element of S, we say that S is 2-local reflexive in $\mathcal{M}(\mathcal{X})$.

Molnár [13] mentioned the problem whether the group of all surjective isometires is 2-local reflexive or not. Even for $C(X)$ for a first countable compact Hausdorff space X, in particular for $C[0, 1]$, the problem seems not be easy. Molnár has already proved among other interesting results that the group of all surjective isometries on $B(H)$ for a separable Hilbert space is 2-local reflexive [14]. In general we may consider

Problem 3. *Under which condition is S 2-local reflexive in $\mathcal{M}(\mathcal{X})$?*

Instead of "2", "Many"-local Maps can be considered: $S \subset \mathcal{M}(\mathcal{X})$;

- ∞-local map : if for all $x \in \mathcal{X}$ there exists $\mathcal{T} \in S$ such that
  $T(x) = \mathcal{T}(x), \qquad x \in \mathcal{X}$.
- 1-local map :
  If S contains a surjection, then
  any $T \in \mathcal{M}(\mathcal{X})$ is 1-local in S!
- $X = \mathbb{R}$ and $S =$ the set of all affine maps.
  Even if $T \in \mathcal{M}(\mathcal{X})$ is 2-local, T need not be affine.
  If $T \in \mathcal{M}(\mathcal{X})$ is 3-local, then $T \in S$

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"2" is interesting. Some how it avoids the triviality. Molnár [12] studied 2-local *complex-linear* surjective isometries of some operator algebras; $S =$ the set of all *complex-linear* surjective isometries on some operator algebras. Recently 2-local *complex-linear* surjective isometries on certain spaces of continuous functions are studied by several authors [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12].

The difficulity of the problem of Molnár seems to depend on the number of the parameters is relatively large. In fact, If $U : C[0,1] \to C[0,1]$ is a surjective isometry, then

$$U(f) = U(0) + \alpha f \circ \varphi, \quad f \in C[0,1],$$
$$U(f) = U(0) + \alpha \overline{f \circ \varphi}, \quad f \in C[0,1].$$

Hence the number of the parameter describing a surjective isometry on $C[0,1]$ is four. We study 2-local sujective isometries on some spaces of complex-valued continuous functions on the closed interval $[0,1]$. We denote by $C^1[0,1]$ the Banach algebra of all continuously differentiable functions on the closed unit interval $[0,1]$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ for $f \in C^1[0,1]$. The following is proved by Miura and Takagi [10].

Theorem 4 (Miura and Takagi). *Let $U : C^1[0,1] \to C^1[0,1]$ be a surjective isometry. Then there exists a constant α of modulus 1 such that one of the following holds.*

(1) $U(f)(t) = U(0)(t) + \alpha f(t), \qquad \forall f \in C^1[0,1],\ \forall t \in [0,1],$
(2) $U(f)(t) = U(0)(t) + \alpha f(1-t), \quad \forall f \in C^1[0,1],\ \forall t \in [0,1],$
(3) $U(f)(t) = U(0)(t) + \alpha \overline{f(t)}, \quad \forall f \in C^1[0,1],\ \forall t \in [0,1],$
(4) $U(f)(t) = U(0)(t) + \alpha \overline{f(1-t)}, \quad \forall f \in C^1[0,1],\ \forall t \in [0,1].$

The group of all surjective isometries on $C^1[0,1]$ is denoted by $\mathrm{Iso}(C^1[0,1])$.

Theorem 5 ([5])**.** *The group* $\mathrm{Iso}(C^1[0,1])$ *is* 2*-local reflexive in* $M(C^1[0,1])$.

Suppose that $T \in M(C^1[0,1])$ is 2-local in $\mathrm{Iso}(C^1[0,1])$. Put $T_0 = T - T(0)$. By the definition T_0 is also 2-local in $\mathrm{Iso}(C^1[0,1])$. We have the following.

Lemma 6. *$T_0(\mathbb{C}) \subset \mathbb{C}$, and $T_0|_{\mathbb{C}}$ is a real-linear isometry on $\mathbb{C}$.*

Hence there exists a complex number α of modulus 1 such that

$$T_0(z) = \alpha z \ (z \in \mathbb{C}) \text{ or } T_0(z) = \alpha \bar{z} \ (z \in \mathbb{C}).$$

The point is to consider the set

$$W = \{f \in C^1[0,1] : \text{If } U(f([0,1])) = f([0,1]) \text{ for an isometry on } \mathbb{C}\text{,then } U \text{ is the identity}\}.$$

Note that : $U(z) = \lambda + \alpha z \ (z \in \mathbb{C})$ or $U(z) = \lambda + \alpha \bar{z} \ (z \in \mathbb{C})$. Put

$$P = \{p + iq : p \text{ and } q \text{ are polynomials of the real coeficients}\}.$$

Many polynomials such as $t + it^2$ are in W, but it is not always the case $(t-1/2)^3 + i(t-1/2)^2 \notin W$. As is expected we have the following.

Lemma 7. *$P \subset \overline{W}$, the uniform closure of W. Hence W is uniformly dense in $C^1[0,1]$.*

Let

$$w(t) = \begin{cases} 0, & t = 0 \\ t^3 \sin \frac{1}{t}, & 0 < t \leq 1 \end{cases}$$

For $f = p + iq \in P$ and $m \in \mathbb{N}$, put

$$f_m = \begin{cases} iw(\frac{1}{m} - t) + \left(p'\left(\frac{1}{m}\right) + iq'\left(\frac{1}{m}\right)\right)\left(t - \frac{1}{m}\right) + p\left(\frac{1}{m}\right) + iq\left(\frac{1}{m}\right), & 0 \leq t \leq \frac{1}{m} \\ p(t) + iq(t), & \frac{1}{m} \leq t \leq 1 \end{cases}$$

Then

$$\{f_m : f = p + iq \in W,\ p \text{ is not constant and } p,\ q,\ 1 \text{ is l.i.}\} \subset W.$$

Lemma 8. *Suppose that $T_0(z) = \alpha z$ $(z \in \mathbb{C})$. Then*

$$T_0(f)(t) = \alpha f(t) \text{ or } T_0(f)(t) = \alpha f(1-t) \text{ for } f \in W.$$

Suppose that $T_0(z) = \alpha \bar{z}$ $(z \in \mathbb{C})$. Then

$$T_0(f)(t) = \alpha \overline{f(t)} \text{ or } T_0(f)(t) = \alpha \overline{f(1-t)} \text{ for } f \in W.$$

Applying Lemma 8 we can deduce the number of the parameters for a 2-local map. Then we have Theorem 5.

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